Electric Circuits
Tutorial Problems

International Faculty of Engineering
Łódź, 2010
DC analysis

**Question 1**

Find the total resistance $R_{AB}$ of the one port AB shown in Fig. 1.

![Fig. 1](image1)

Compute the currents $i_6$ and $i_7$ in the circuit of Fig. 2.

**Data:**

$R_1 = R_3 = 10 \, \Omega$, $R_2 = R_4 = R_5 = 20 \, \Omega$, $R_6 = 15 \, \Omega$, $R_7 = 30 \, \Omega$, $V_S = 60 \, V$.

**Solution**

To find the total resistance of the one-port AB we replace the $\Delta$-connection consisting of the resistors $R_2, R_3, R_5$ by $Y$-connection (see Fig. 3)

![Fig. 3](image2)

The resistances $R_{23}, R_{25}, R_{35}$ are as follows:

$$R_{23} = \frac{R_2 R_3}{R_2 + R_3 + R_5} = 4\Omega$$

$$R_{25} = \frac{R_2 R_5}{R_2 + R_3 + R_5} = 8\Omega$$
\[ R_{35} = \frac{R_3 R_5}{R_2 + R_3 + R_5} = 4 \Omega \]

Hence, the total resistance is
\[ R_{AB} = R_{35} + \frac{(R_1 + R_{23})(R_4 + R_{25})}{R_1 + R_{23} + R_4 + R_{25}} = 13.33 \Omega \]

Now we consider the circuit of Fig. 2 repeated below.

![Circuit Diagram](image)

We compute the resistance faced by the voltage source
\[ R = R_{AB} + \frac{R_6 R_7}{R_6 + R_7} \]

and the current flowing through the source
\[ i_{AB} = \frac{V_S}{R_{AB} + \frac{R_6 R_7}{R_6 + R_7}} = \frac{60}{23.33} = 2.57 \text{A} \]

Since the resistors \( R_6 \) and \( R_7 \) are connected in parallel, then
\[ i_6 = i_{AB} \frac{R_7}{R_6 + R_7} = 1.72 \text{A} \]

Current \( i_7 \) can be determined using KCL:
\[ i_7 = -i_{AB} + i_6 = -0.85 \text{A} \]

**Question 2**

For the one-port AB shown in Fig. 3 determine Thevenin’s circuit.

(i) Compute the current flowing through the resistor \( R = 5 \Omega \) connected to terminals A and B.
(ii) Compute the voltage between terminals A and B when the parallel combination of \( R_0 = 6 \Omega \) and \( I_0 = 1 \text{A} \) is connected to terminals AB.
Data:
\( R_1 = R_2 = 12\,\Omega \), \( R_3 = 6\,\Omega \), \( E_1 = 6\,V \), \( E_2 = 3\,V \)

**Solution**

To determine Thevenin’s circuit for the one-port AB we need voltage \( v_{0C} \) between terminals A, B (see Fig.6) and the input resistance \( R_{eq} \) of the one-port after setting to zero the source voltages (see Fig.7).

\[
\begin{align*}
\left(R_1 + R_2 + R_3\right)i - E_1 + E_2 &= 0 \\
i &= \frac{E_1 - E_2}{R_1 + R_2 + R_3} = 0.1\,A
\end{align*}
\]

Now we can determine \( v_{0C} \) using KVL in the closed node sequence ACBA and Ohm’s law:

\[
\begin{align*}
-E_2 - R_2i + v_{0C} &= 0 \\
v_{0C} &= E_2 + R_2i = 4.2\,V
\end{align*}
\]

The total resistance of the one-port shown in Fig.7 is given by

\[
R_{eq} = \frac{\left(R_1 + R_3\right)R_2}{R_1 + R_3 + R_2} = 7.2\,\Omega
\]
The Thevenin circuit is depicted in Fig. 8

![Fig. 8](image1)

\[ v_{0C} = 4.2 \text{V} \]
\[ R_{eq} = 7.2 \Omega \]

![Fig. 9](image2)

\[ v_{0C} = 4.2 \text{V} \]
\[ R_{eq} = 7.2 \Omega \]

(i) When the one-port AB is terminated by the resistor \( R = 5 \Omega \), the current \( i_R \) flowing through this resistor can be computed using Thevenin’s circuit (see Fig. 9). Hence, we find

\[ i_R = \frac{v_{0C}}{R_{eq} + R} = 0.34 \text{A} \]

(ii) Now we connect to the one-port AB the parallel connection of \( R_0 = 6 \Omega \) and \( I_0 = 1 \text{A} \) and replace the one-port by Thevenin’s circuit. As a result we obtain the circuit shown in Fig. 10

![Fig. 10](image3)

Let us replace the parallel connection of \( R_0 \) and \( I_0 \) by an equivalent series connection of the same resistor \( R_0 \) and a voltage source \( E_0 = R_0 I_0 = 6 \text{V} \) (see Fig. 11).

![Fig. 11](image4)
Next we compute current \( i' \) using KVL and Ohm’s law:

\[
(R_{eq} + R_0)i' - v_{OC} - E_0 = 0
\]

\[
i' = \frac{v_{OC} + E_0}{R_{eq} + R_0} = \frac{4.2 + 6}{7.2 + 6} = 0.77 \text{ A}.
\]

We apply KVL, to the closed node sequence ACBA, and Ohm’s law:

\[-R_0i' + E_0 + v_{AB} = 0\]

and find voltage \( v_{AB} \)

\[
v_{AB} = R_0i' - E_0 = 6 \cdot 0.77 - 6 = 1.38 \text{ V}
\]

**Question 3**

Describe the circuit shown in Fig. 12 using the node method. Compute the currents \( i_2, i_3 \) and the power consumed by the resistor \( R_3 \).

![Fig. 12](image)

**Data:**

\( R_1 = 20 \ \Omega, \quad R_2 = 30 \ \Omega, \quad R_3 = 10 \ \Omega, \quad R_4 = 10 \ \Omega, \quad I_{s_1} = 1 \ \text{A}, \quad I_{s_2} = 4 \ \text{A}. \)

**Solution**

To write the node equations we introduce the node voltages \( e_1, e_2, e_3 \) as shown in Fig. 13
The set of the node equations is as follows:

\[
\begin{align*}
\frac{e_1}{R_4} + \frac{e_1 - e_2}{R_1} &= -I_{S_2}, \\
\frac{e_2 - e_1}{R_1} + \frac{e_2 - e_3}{R_2} &= I_{S_1}, \\
\frac{e_3 - e_2}{R_2} + \frac{e_3}{R_3} &= I_{S_1}.
\end{align*}
\]

Substituting the resistances we obtain

\[
\begin{align*}
\frac{e_1}{10} + \frac{e_1 - e_2}{20} &= -4, \\
\frac{e_2 - e_1}{20} + \frac{e_2 - e_3}{30} &= 1, \\
\frac{e_3 - e_2}{30} + \frac{e_3}{10} &= 4.
\end{align*}
\]

We solve the set of these equations using the substituting method. As a result we find

\[
\begin{align*}
e_1 &= -22.86 \text{ V}, \\
e_2 &= 11.42 \text{ V}, \\
e_3 &= 32.85 \text{ V}.
\end{align*}
\]

Next we express currents \(i_2\) and \(i_3\) in terms of the node voltages

\[
\begin{align*}
i_2 &= \frac{e_2 - e_3}{R_2} = -0.71 \text{ A}, \\
i_3 &= \frac{e_3}{R_3} = 3.29 \text{ A}.
\end{align*}
\]

Power consumed by the resistor \(R_3\) is
\[ P_1 = R_3 i_3^2 = 107.9 \text{ W}. \]

**Question 4**

In the circuit of Fig. 14 replace the \( \text{Y} \)-connected circuit consisting of resistors \( R_2, R_3, R_4 \) by an equivalent \( \Delta \)-connected circuit. For the one-port AB determine Thevenin’s and Norton’s circuits.

![Fig. 14](image)

Data: \( R_1 = 2 \Omega, \quad R_2 = R_3 = 1 \Omega, \quad R_4 = 4 \Omega, \quad R_5 = 3 \Omega, \quad I_s = 1 \text{ A}. \)

**Solution**

The circuit after transformation of the \( \text{Y} \)-connected circuit into \( \Delta \)-connected circuit is shown in Fig. 15.

![Fig. 15](image)

The resistances \( R_{24}, R_{23}, R_{34} \) are as follows:

\[
R_{24} = R_2 + R_4 + \frac{R_2 R_4}{R_3} = 9 \Omega, \\
R_{23} = R_2 + R_3 + \frac{R_2 R_3}{R_4} = 2.25 \Omega, \\
R_{34} = R_3 + R_4 + \frac{R_3 R_4}{R_2} = 9 \Omega.
\]
To determine Thevenin’s circuit for the one-port AB we need the total resistance $R_{eq}$ of the one-port after the current source $I_S$ is set to zero (see Fig. 16) and the voltage $V_{OC}$, between open circuited terminals A,B, as indicated in Fig. 15.

![Fig. 16](image1)

The circuit of Fig. 16 can be reduced as depicted in Fig. 17, where

$$R_{123} = \frac{R_1R_{23}}{R_1 + R_{23}} = 1.06\Omega,$$

$$R_{345} = \frac{R_{34}R_5}{R_{34} + R_5} = 2.25\Omega.$$

![Fig. 17](image2)

Hence, we have

$$R_{eq} = \frac{(R_{24} + R_{123})R_{345}}{R_{24} + R_{123} + R_{345}} = 1.84\Omega.$$

In order to find $V_{OC}$ we rearrange the circuit of Fig. 15 as shown in Fig. 18.

![Fig. 18](image3)
In this circuit the series connection of $R_{24}$ and $E_S$ replaces the parallel connection of $R_{24}$ and $I_S$ appeared in Fig. 15. Hence, it holds

$$E_S = I_S R_{24} = 9 \text{ V}.$$  

We compute the current $i$

$$i = \frac{E_S}{R_{24} + R_{123} + R_{345}} = 0.73 \text{ A}$$

and the voltage $V_{OC}$

$$V_{OC} = R_{345} i = 1.65 \text{ V}.$$  

Thus, the Thevenin circuit is as shown in Fig. 19

![Fig. 19](image)

Norton’s circuit is depicted in Fig. 20, where

$$I_{sc} = \frac{E_{OC}}{R_{eq}} = 0.90 \text{ A}.$$  

![Fig. 20](image)

**Question 5**

In the circuit shown in Fig. 21 current $i_2$ is given, $i_2 = 1.5 \text{ A}$. 

(i) Compute the source voltage $V_S$ and the power consumed by the resistor $R_4$. 

(ii) Compute new current $i'_2$, flowing through $R_2$, when the resistor $R_4$ is short circuited.
Data:
\[ R_1 = 10 \, \Omega, \quad R_2 = 20 \, \Omega, \quad R_3 = 15 \, \Omega, \quad R_4 = 5 \, \Omega. \]

**Solution**

(i) At first we compute the voltage \( v_2 \) across the \( R_2 \):
\[
v_2 = R_2 i_2 = 30 \, \text{V}.
\]
Since \( R_3 \) and \( R_4 \) are connected in series and the voltage across this connection is \( v_2 \) then
\[
i_3 = \frac{v_2}{R_3 + R_4} = 1.5 \, \text{A}.
\]
Using KCL at the top node we have
\[
i_1 = i_2 + i_3 = 3 \, \text{A}.
\]
To find \( V_S \) we apply KVL in the loop consisting of \( V_S, R_1 \) and \( R_2 \)
\[
V_S = R_1 i_1 + R_2 i_2 = 60 \, \text{V}.
\]
The power consumed by the resistor \( R_4 \) is
\[
P_4 = R_4 i_3^2 = 11.25 \, \text{W}.
\]

(ii) The circuit with the short circuited resistor \( R_4 \) is shown in Fig. 22.
In this circuit we find:

\[ i_1 = \frac{V_5}{R_1 + \frac{R_2 R_3}{R_2 + R_3}} = \frac{60}{10 + \frac{20 \cdot 15}{35}} = 3.23 \text{ A}, \]

\[ i_2 = i_1 \frac{R_3}{R_2 + R_3} = 3.23 \frac{15}{35} = 1.38 \text{ A}. \]

**Question 6**

In the circuit shown in Fig. 23 compute the current \( i_R \) flowing through the resistor \( R \) in three cases:

(i) \( R = 2.5 \text{ } \Omega \)
(ii) \( R = 7.5 \text{ } \Omega \)
(iii) \( R = 20 \text{ } \Omega \)

Apply Thevenin’s theorem.

![Fig. 23](image)

Data:

\( R_1 = 10 \text{ } \Omega \), \( R_2 = 15 \text{ } \Omega \), \( R_3 = 25 \text{ } \Omega \), \( E_3 = 5 \text{ } \text{V} \), \( I_0 = 1 \text{ } \text{A} \).

**Solution**

We replace the one-port AB, consisting of \( R_1, R_2, R_3, I_0, \) and \( E_3 \), by the equivalent Thevenin circuit. Thus, we need to compute voltage \( v_{OC} \) (see Fig. 24) and the resistance \( R_{eq} \) (see Fig. 25)

![Fig. 24](image)  
![Fig. 25](image)

Let us consider the circuit of Fig. 24 and replace the parallel connection of \( I_0 \) and \( R_1 \) by an equivalent series connection of the voltage source \( E_1 = R_1 I_0 = 10 \text{ } \text{V} \) and the same resistor \( R_1 \) (see Fig. 26).
Since the terminals A,B are left open-circuited, it holds
\[ i = \frac{E_1 - E_3}{R_1 + R_2 + R_3} = \frac{5}{50} = 0.1 \text{A}. \]

Using KVL we obtain
\[ V_{OC} = E_3 + R_3i = 5 + 2.5 = 7.5 \text{V}. \]

In order to find \( R_{eq} \) we consider the circuit depicted in Fig. 25 and compute the input resistance of the one-port AB
\[ R_{eq} = \frac{(R_1 + R_2)R_3}{R_1 + R_2 + R_3} = \frac{25 \cdot 25}{50} = 12.5 \Omega. \]

Using the equivalent Thevenin circuit we replace the circuit of Fig. 23 by the circuit shown in Fig. 27.

In this circuit it holds
\[ i_R = \frac{V_{OC}}{R_{eq} + R}. \]

Hence, we obtain:
(i) \[ i_R = \frac{7.5}{12.5 + 2.5} = 0.5 \text{ A} \]

(ii) \[ i_R = \frac{7.5}{12.5 + 7.5} = 0.375 \text{ A} \]

(iii) \[ i_R = \frac{7.5}{12.5 + 20} = 0.230 \text{ A} \]

**Question 7**

![Fig. 28](image)

(i) In the circuit shown in Fig. 28 determine the branch currents \( i_1, i_2, i_3, i_4 \) using the node method.

(ii) Write the node equations if the current source \( I_{s_1} \) is replaced by series combination of a voltage source \( E = 12 \text{V}, \) directed to the right, and a resistor \( R = 12\Omega \).

Data:
\[ R_1 = 10 \Omega, \quad R_2 = R_3 = 20 \Omega, \quad R_4 = 40 \Omega, \quad I_{s_1} = 1\text{A}, \quad I_{s_2} = 3\text{A}. \]

**Solution**

To write the node equations we choose the bottom node as the datum node and introduce the node voltages \( e_1 \) and \( e_2 \) as shown in Fig. 29

![Fig. 29](image)
(i) The node equations have the form

\[
\frac{e_1}{R_1} + \frac{e_2}{R_2} + \frac{e_1 - e_2}{R_4} = -I_{s_1},
\]
\[
\frac{e_2 - e_1}{R_4} + \frac{e_2}{R_3} = I_{s_1} + I_{s_2}.
\]

We substitute the resistance and source values and perform simple rearrangements. As a result we obtain the following set of equations in two unknown variables \( e_1 \) and \( e_2 \):

\[-e_2 = -407e_1,\]
\[-e_1 + 3e_2 = 160.\]

This set of equations is solved using the substituting method. As a result we obtain:

\[e_1 = 2V, \quad e_2 = 54V.\]

Using the node voltages we compute the branch currents as follows:

\[i_1 = \frac{e_1}{R_1} = 0.2\ A, \quad i_2 = \frac{e_1}{R_2} = 0.1\ A, \quad i_3 = \frac{e_2}{R_3} = 2.7\ A, \quad i_4 = \frac{e_2 - e_1}{R_4} = 1.3\ A.\]

(ii) The circuit with a new branch consisting of the resistor \( R \) and the voltage source \( E \) is shown in Fig. 30.

Current \( i \) flowing through this branch is given by

\[i = \frac{e_1 - e_2 + E}{R}.\]

Hence, the node equations are

\[e_1 \left(\frac{1}{R_1} + \frac{1}{R_2}\right) + \frac{e_1 - e_2}{R_4} + \frac{e_1 - e_2 + E}{R} = 0,\]
\[ \frac{e_2 - e_1}{R_4} + \frac{e_2 - e_1 - E}{R} + \frac{e_2}{R_3} = I_s. \]

Setting the resistance and source values we obtain:

\[ 31e_1 - 13e_2 = -120, \]
\[ -13e_1 + 19e_2 = 480. \]

**Question 8**

(iv) For the one-port AB, shown in Fig. 31, determine Thevenin’s and Norton’s circuits.

(v) Find the current flowing through the resistor \( R = 3 \, \Omega \) connected to the terminals A and B. Compute the power consumed by this resistor and voltage across the resistor \( R_3 \).

Data:

\[ R_1 = R_2 = 5 \, \Omega, \quad R_3 = 10 \, \Omega, \quad R_4 = 2 \, \Omega, \quad E = 10V, \quad I = 1A. \]

**Solution**

(i) Thevenin’s and Norton’s circuits are shown in Fig. 32.

To find \( R_{eq} \) we consider the circuit after the voltage source is short-circuited and the current source is open-circuited (see Fig. 33)
Total impute resistance $R_{eq}$ of this one-port is

$$R_{eq} = R_4 + \frac{R_3(R_1 + R_2)}{R_1 + R_2 + R_3} = 7 \Omega.$$  

$V_{0C}$ is the voltage between the open-circuited terminals A, B as shown in Fig. 34.

To find $V_{0C}$ we transform the one-port AB as depicted in Figs. 35 – 37.
The circuit of Fig. 36 shows a current $i = \frac{E}{R_l} + I = 3$ A.

In the circuit of Fig. 37 current $i_3$ traverses the resistors $R_3, R_2, R_1$ and the voltage source $\hat{E}$. Hence, it holds

$$i_3 = \frac{\hat{E}}{R_1 + R_2 + R_3} = 0.75 \text{ A}.$$ 

Since the voltage across the resistor $R_4$ is zero, the voltage $V_{OC}$ is the same as $V_3 = R_3 i_3$. Thus, we have

$$V_{OC} = R_3 i_3 = 7.5 \text{ V}.$$ 

(iii) To determine current flowing through the resistor $R$ connected to terminals A and B we replace the one-port AB by Thevenin’s circuit (see Fig. 38).
Using KVL and Ohm’s law we obtain

\[ i = \frac{V_{oc}}{R_{eq} + R} = 0.75 \text{A} . \]

Hence, the power consumed by \( R \) is

\[ P = Ri^2 = 1.687 \text{W} . \]

To find voltage \( \hat{v}_3 \) across resistor \( R_3 \) we consider the one-port AB terminated by the resistor \( R \) (see Fig. 39)

KVL and Ohm’s law lead to the relationship

\[ \hat{v}_3 = i(R_4 + R) = 3.75 \text{V} . \]

**Question 9**

In the circuit shown in Fig. 40 the power consumed by the resistor \( R_5 \) is \( P_5 = 10 \text{W} \).

(i) Compute the source current \( I_S \).

(ii) Replace the resistor \( R_5 \) by resistor \( R'_5 = 30\Omega \) and compute the power consumed by this resistor.

Data:

\[ R_1 = R_2 = R_3 = 20\Omega, \quad R_4 = R_5 = 10\Omega. \]
Solution  
(i) Since the same current traverses resistors $R_4$ and $R_5$ then the power consumed by resistor $R_5$ is given by the equation

$$P_5 = R_5 i_4^2.$$  

Hence, we find

$$i_4 = \frac{P_5}{\sqrt{R_5}} = 1\text{A}.$$  

Having $i_4$ we compute the voltage between the nodes C and D and next current $i_3$

$$v_{CD} = (R_4 + R_5) i_4 = 20\text{V}$$

$$i_3 = \frac{v_{CD}}{R_3} = 1\text{A}.$$  

Applying KCL at node C we obtain

$$i_2 = i_3 + i_4 = 2\text{A}.$$  

Hence, we have

$$v_{AC} = R_2 i_2 = 40\text{V}.$$  

KVL applied to the loop ACDBA leads to the equation

$$v_{AB} = v_{AC} + v_{CD} = 60\text{V}.$$  

Consequently, it holds

$$i_1 = \frac{v_{AB}}{R_1} = 3\text{A}.$$  

Using KCL at node A we have

$$I_S = i_1 + i_2 = 5\text{A}.$$  

(ii) We now consider the circuit shown in Fig. 41a and reduce it as depicted in Figs. 41b and c.

![Fig. 41a](image_url)

Thus, the total resistor faced by the current source is $R_T = 12.5\text{Ω}$. Hence, using successively circuits of Figs 41c, 41b, and 41a we find
\[ v'_{AB} = 12.5 \cdot 5 = 62.5 \text{ V}, \]
\[ v'_{CD} = \frac{v'_{AB}}{20 + 13.33} \cdot 13.33 = 25 \text{ V}, \]
\[ i'_4 = \frac{v'_{CD}}{10 + 30} = 0.625 \text{ A}. \]

Since the same current traverses resistors \( R_4 \) and \( R'_5 \) we obtain
\[ P'_5 = R'_5 (i'_4)^2 = 11.72 \text{ W}. \]

Question 10
(i) Replace the one-port AB shown in Fig. 42 by the equivalent Thevenin circuit.
(ii) Compute the current flowing through the resistor \( R = 2 \Omega \) connected to terminals A and B.
(iii) Determine the resistance of a resistor connected to terminals A and B knowing that the power consumed by this resistor is 0.35W.

Data:
\( R_1 = 2 \Omega, \ R_2 = 3 \Omega, \ R_3 = 2 \Omega, \ R_4 = 4 \Omega, \ E_1 = 1 \text{ V}, \ E_2 = 3 \text{ V}. \)
Solution

(i) The equivalent Thevenin circuit consists of a voltage source $V_{OC}$ and a resistor $R_{eq}$ connected in series. To find $R_{eq}$ we consider the circuit shown in Fig. 43 obtained from the original circuit by setting $E_1 = 0$ and $E_2 = 0$.

![Fig. 43](image)

Resistance $R_{eq}$ is as follows:

$$R_{eq} = \left( \frac{R_1R_3}{R_1 + R_3} + R_4 \right) \frac{R_2}{R_2} = \frac{R_1R_3}{R_1 + R_3 + R_4 + R_2} = 1.875 \Omega.$$  

$V_{OC}$ is the voltage between terminals A and B that are left open-circuited (see Fig. 44). It can be computed using the node method.

![Fig. 44](image)

The node equations are as follows:

$$\frac{e_1}{R_3} + \frac{e_1 - E_1}{R_1} + \frac{e_1 - e_2}{R_4} = 0,$$

$$\frac{e_2 - e_1}{R_4} + \frac{e_2 - E_2}{R_2} = 0.$$
We set the values of the resistances and voltage sources

\[
\frac{e_1}{2} + \frac{e_1 - 1}{2} + \frac{e_1 - e_2}{4} = 0, \\
\frac{e_2 - e_1}{4} + \frac{e_2 - 3}{3} = 0.
\]

To solve this set of equations we use the substituting method. After eliminating \(e_1\) we obtain

\[
\frac{32}{3} e_2 = 22 \Rightarrow e_2 = 2.06 \text{ V}.
\]

Note that

\(V_{OC} = e_2 = 2.06 \text{ V}.\)

(ii) To find the current flowing through the resistor \(R\) we exploit the equivalent Thevenin circuit as shown in Fig. 45.

![Fig. 45](image)

Using KVL we obtain

\[
i = \frac{V_{OC}}{R_{eq} + R} = \frac{2.06}{1.875 + 2} = 0.53 \text{ A}.
\]

(iii) We now consider the circuit of Fig. 46 where \(R_x\) is an unknown variable.

![Fig. 46](image)
The power consumed by the resistor $R_x$ is

$$P_x = 0.35 = R_x i_x^2,$$

where

$$i_x = \frac{V_{OC}}{R_{eq} + R_x} = \frac{2.06}{1.875 + R_x}.$$

Hence, we can write the equation

$$0.35 = R_x \left( \frac{2.06}{1.875 + R_x} \right)^2,$$

which, after some rearrangements becomes

$$R_x^2 - 8.374R_x + 3.514 = 0.$$

Solving this equations we obtain $R_{x_1} = 7.931\Omega$, $R_{x_2} = 0.443\Omega$. 
**Transient analysis**

**Question 1**
The switch in the circuit shown in Fig. 1 is close until the steady state prevails and then it is opened. Assuming that the opening occurs at \( t = 0 \) find the current \( i_L(t) \) and the voltage \( v_L(t) \).

![Circuit Diagram](image)

**Data:** \( E = 100 \text{V}, \ R_1 = 1 \Omega, \ R_2 = 8.1 \Omega, \ R_3 = 9 \Omega, \ L = 0.1 \text{H} \).

**Solution**
When the switch is closed, the circuit is in the steady state, all currents are constant and the voltage across the inductor, given by \( v_L = L \frac{di_L}{dt} \), is equal to zero. Thus, the inductor can be replaced by a short circuit and the circuit becomes as shown in Fig. 2.

![Circuit Diagram](image)

In this circuit it holds

\[
i_L(0^-) = \frac{E}{R_1 + \frac{R_2 R_3}{R_2 + R_3}} \cdot \frac{R_2}{R_2 + R_3} = 9 \text{A}.
\]

Since the inductor current \( i_L(t) \) satisfies the continuity property \( i_L(0) = i_L(0^-) = 9 \text{A} \). When the switch is open the circuit has the form shown in Fig. 3.
The current \( i_L(t) \) is specified by the equation

\[
    i_L(t) = i_L(\infty) + (i_L(0) - i_L(\infty))e^{-\frac{t}{\tau}},
\]

where

\[
    \tau = \frac{L}{R_1 + R_3} = 0.01 \text{s}.
\]

The steady state current \( i_L(\infty) \) is

\[
    i_L(\infty) = \frac{E}{R_1 + R_3} = 10 \text{A}.
\]

Substituting \( i_L(0) = 9 \text{A}, \ i_L(\infty) = 10 \text{A} \) and \( \tau = 0.01 \text{s} \) into the equation (2) yields

\[
    i_L(t) = (10 - e^{-100t}) \text{A}.
\]

Next we find \( v_L(t) \)

\[
    v_L(t) = L \frac{di_L}{dt} = 0.1 \cdot 100 e^{-100t} = 10 e^{-100t}.
\]

**Question 2**

The switch in the circuit shown in Fig. 4 is close until the steady state prevails and then it is opened. Assuming that the opening occurs at \( t = 0 \) find the current \( i_C(t) \).

Data:

\[
    E = 120 \text{ V}, \quad R_1 = 10 \Omega, \quad R_2 = 100 \Omega, \quad R_3 = 20 \Omega, \quad C = 10 \mu\text{F}.
\]
**Solution**

When the switch is closed, the circuit is in the steady state, the current \( i_c = C \frac{dv_c}{dt} = 0 \). Thus, the capacitor can be replaced by an open circuit (see Fig. 5).

![Fig. 5](image-url)

In the resistive circuit shown in Fig. 5 we find

\[
v_c(0^-) = i(0^-)R_3 = \frac{E}{R_1 + R_3} - R_3 = 80\text{V}. \tag{7}
\]

Since the voltage \( v_c(t) \) across the capacitor satisfies the continuity property, \( v_c(0) = v_c(0^-) = 80\text{V} \). When the switch is open the circuit has the form shown in Fig. 6.

![Fig. 6](image-url)

In this circuit it holds

\[
v_c(t) = v_c(\infty) + (v_c(0) - v_c(\infty))e^{-t/\tau}, \tag{8}
\]

where

\[
\tau = (R_2 + R_3)C = 120 \cdot 10^{-3} = 1.2 \cdot 10^{-3}\text{s}. \tag{9}
\]

The steady state voltage \( v_c(\infty) = 0 \). Substituting \( v_c(0) = 80\text{V} \), \( v_c(\infty) = 0 \), and \( \tau = 1.2 \cdot 10^{-3} \) into the equation (8) yields

\[
v_c(t) = 80e^{-t/(1.2 \cdot 10^{-3})}. \tag{10}
\]

Hence, we find

\[
i_c(t) = C \frac{dv_c(t)}{dt} = 10^{-5} \cdot 80 \left( \frac{1}{1.2 \cdot 10^{-3}} e^{-t/(1.2 \cdot 10^{-3})} \right) = \left( -\frac{2}{3} e^{-t/(1.2 \cdot 10^{-3})} \right) \text{A}. \tag{11}
\]
**Question 3**

The switch in the circuit shown in Fig. 7 is close until the steady state prevails and then it is opened. Assuming that the opening occurs at \( t = 0 \) find \( v_c(t) \), \( i_L(t) \) and the voltage \( v(t) \) across the switch.

![Fig. 7](image)

**Solution**

When the switch is close, the circuit is in the steady state, hence, \( i_C = 0 \) and \( v_L = 0 \). Consequently, the circuit has the form shown Fig. 8.

![Fig. 8](image)

In the circuit depicted in Fig. 8 we write

\[
    i_L(0^-) = \frac{E}{R_1} = 1\text{A} = i_L(0),
\]
\[
    v_C(0^-) = 0 = v_C(0).
\]

When the switch is open the circuit consists of two parts which can be analysed separately. They are shown in Figs. 9 and 10.

![Fig. 9](image)  
![Fig. 10](image)

Voltage \( v_C(t) \) in the circuit depicted in Fig. 9 is given by the equation
\[ v_c(t) = v_c(\infty) + (v_c(0) - v_c(\infty))e^{-\frac{t}{\tau_1}}, \]  
\[ \tau_1 = R_1C = 0.4 \cdot 10^{-3}\, \text{s}, \]
\[ v_c(\infty) = E = 100\, \text{V}. \]
Since \[ v_c(0) = 0, \]
we have
\[ v_c(t) = 100 \left( 1 - e^{-\frac{t}{0.4 \cdot 10^{-3}}} \right) \, \text{V}. \]  

The circuit shown in Fig. 10 is described by the equation
\[ i_L(t) = i_L(\infty) + (i_L(0) - i_L(\infty))e^{-\frac{t}{\tau_2}}, \]  
where
\[ \tau_2 = \frac{L}{R_2} = 2 \cdot 10^{-3}\, \text{s}, \]
\[ i_L(\infty) = 0. \]
Since \[ i_L(0) = 1\, \text{A} \]
then
\[ i_L(t) = e^{-\frac{t}{2 \cdot 10^{-3}}} \, \text{A}. \]  

To find the voltage \[ v(t), \]
across the switch we apply KVL in the circuit shown in Fig. 7
\[ v(t) = v_c(t) - v_L(t) \]
where
\[ v_L(t) = L \frac{di_L(t)}{dt} = 0.2 \left( -\frac{1}{2 \cdot 10^{-3}} \right) e^{-\frac{t}{2 \cdot 10^{-3}}} = -100 e^{-\frac{t}{2 \cdot 10^{-3}}} \, \text{V}. \]  
Substituting (13) and (17) into (16) yields
\[ v(t) = \left( 100 \left( 1 - e^{-\frac{t}{0.4 \cdot 10^{-3}}} \right) + 100 e^{-\frac{t}{2 \cdot 10^{-3}}} \right) \, \text{V}. \]

**Question 4**

The switch in the circuit depicted in Fig. 11 is on the terminal 1 until the steady state prevails. At \[ t = 0 \] it is flipped from terminal 1 to 2. Find the voltage \[ v_{12}(t) \] (between the terminals 1 and 2) and the energy consumed by the resistor \[ R_4 \] from \[ t = 0 \] to \[ t \to \infty \].

**Data:**
\[ R_1 = 10\, \Omega \]
\[ R_2 = R_3 = 20\, \Omega \]
\[ R_4 = 10\, \Omega \]
\[ L = 0.1\, \text{H} \]
\[ C = 10\, \mu\text{F} \]
\[ E = 2\, \text{V} \].
Solution
When the switch is on the terminal 1 and the circuit is in the steady state the inductor $L$ can be replaced by a short circuit and the capacitor $C$ by an open circuit (see Fig. 12).

![Fig. 12](image)

In this circuit we find

$$i_L(0^-) = \frac{E}{R_1 + \frac{R_2 R_4}{R_2 + R_4}} = 0.12 \text{A} = i_L(0),$$

$$v_C(0^-) = i_L(0^-) \frac{R_2}{R_2 + R_4} = 0.8 \text{V} = v_C(0).$$

When the switch is on the terminal 2 the circuit is separated into two parts (see Fig.13).

![Fig. 13](image)

To find $i_L(t)$ we rearrange the left part of the circuit as shown in Fig. 14.

![Fig. 14](image)
In this circuit

\[ i_L(t) = i_L(\infty) + (i_L(0) - i_L(\infty))e^{-\frac{t}{\tau_1}}, \]  \hspace{1cm} (18)

where

\[ \tau_1 = \frac{L}{R_{123}} = \frac{0.1}{20} = 0.005 \text{s}, \quad i_L(\infty) = \frac{E}{R_{123}} = \frac{2}{20} = 0.1 \text{A}. \]

Since \( i_L(0) = 0.12 \), then

\[ i_L(t) = \left(0.1 + 0.02e^{-\frac{t}{0.005}}\right) \text{A}. \] \hspace{1cm} (19)

To find \( v_C(t) \) we consider the right part of the circuit, consisting of the resistor \( R_4 \) and the capacitor \( C \) connected in parallel. We write

\[ v_C(t) = v_C(\infty) + (v_C(0) - v_C(\infty))e^{-\frac{t}{\tau_2}}, \] \hspace{1cm} (20)

where

\[ \tau_2 = R_4C = 10^{-4} \text{s}, \quad v_C(\infty) = 0. \]

Hence, we have

\[ v_C(t) = \left(0.8e^{-\frac{t}{10^{-4}}}\right) \text{V}. \] \hspace{1cm} (21)

In order to find \( v_{12}(t) \) we apply KVL

\[ v_{12}(t) = v_2(t) - v_C(t), \] \hspace{1cm} (22)

where

\[ v_2(t) = i_L(t) \frac{R_2R_3}{R_2 + R_3} = \left(1 + 0.2e^{-\frac{t}{0.005}}\right) \text{V}. \] \hspace{1cm} (23)

Substituting (21) and (23) into (22) gives

\[ v_{12}(t) = \left(1 + 0.2e^{-\frac{t}{0.005}} - 0.8e^{-\frac{t}{10^{-4}}}\right) \text{V}. \]

The energy consumed by the resistor \( R_4 \) is specified by the equation

\[ W_4 = \int_0^\infty \frac{v_C^2(t)}{R_4} \, dt = 0.1 \int_0^\infty 0.64e^{-\frac{2t}{10^{-4}}} \, dt = 0.064 \left(-\frac{10^{-4}}{2}\right) e^{-\frac{2t}{10^{-4}}} \bigg|_0^\infty = 3.2 \cdot 10^{-6} \text{ J}. \]
For $t < 0$ the switch is on terminal A and the circuit is in steady state. At $t = 0$ the switch is flipped to terminal B. Determine $i_L(t)$, $i_1(t)$ and $v_L(t)$ for $t \geq 0$.

**Solution**

For $t < 0$ the current $i_L(t)$ is constant, hence, it holds

$$v_L(t) = L \frac{di_L(t)}{dt} = 0.$$  \hspace{1cm} (24)

Thus, at $t = 0^-$ the inductor can be replaced by short circuit as shown in Fig. 16.

Since $i_L(0^-) = 0$, we have

$$i_L(0^-) = i_L(0^-) = \frac{10}{2+3} = 2\text{ A}.$$  \hspace{1cm} (25)

On the basis of the continuity property

$$i_L(0) = i_L(0^-) = 2\text{ A}.$$  \hspace{1cm} (26)

When the switch is on terminal B the circuit becomes as shown in Fig. 17.
We take into account the one-port lying inside the dotted rectangle and replace it by Thevenin’s circuit. To find $V_{OC}$ and $R_{eq}$ we create two circuits depicted in Figs. 18 and 19.

Analyzing the circuits we obtain

$$V_{OC} = \frac{20}{12} \times 6 = 10 \text{ V} , \quad R_{eq} = 3 \Omega .$$

Thus, the circuit shown in Fig. 17 is reduced to the one depicted in Fig. 20.

The time constant is given by

$$\tau = \frac{2}{3} \text{ s} .$$

As $t \to \infty$ the current $i(t)$ becomes constant, consequently the voltage across the inductor is equal to zero and the inductor can be replaced by short circuit (see Fig. 21).
Current $i_L(\infty)$ is

$$i_L(\infty) = \frac{10}{3} \text{ A}.$$  \hfill (29)

Now we use the general formula

$$i_L(t) = i_L(\infty) + (i_L(0) - i_L(\infty)) e^{\frac{-t}{\tau}}$$  \hfill (30)

and substitute (26) and (29)

$$i_L(t) = \frac{10}{3} + \left(2 - \frac{10}{3}\right) e^{\frac{-t}{\tau}} = \frac{10}{3} - \frac{4}{3} e^{\frac{-t}{\tau}},$$  \hfill (31)

where $\tau = \frac{2}{3}$ s. The plot of $i_L$ against $t$ is shown in Fig. 22.

To find the current $i(t)$, for $t > 0$ we replace the inductor in Fig. 17 by the current source $i_L(t)$. As a result we obtain the resistive circuit shown in Fig. 23.
We replace the parallel connection of the 6-ohm resistor and the current source \( i_L(t) \) by the series connection of the resistor and a voltage source. This voltage source is \( 6i_L(t) \) (see Fig. 24).

Using KVL and Ohm’s law we obtain

\[
i_1(t) = \frac{20 + 6i_L(t)}{3 + 3 + 6}, \tag{32}\]

and substitute (31) into equation (32)

\[
i_1(t) = \frac{20 + 20 - 8e^{-\frac{t}{\tau}}}{12} = \left(\frac{10}{3} - \frac{2}{3}e^{-\frac{t}{\tau}}\right) A \quad \text{for} \quad t > 0. \tag{33}\]

At \( t = 0^- \) the current \( i_1(0^-) \), flowing in the circuit shown in Fig. 16, equals

\[
i_1(0^-) = \frac{10}{2 + 3} = 2 \text{ A}. \]

To determine the voltage \( v_L \) we use the formula

\[
v_L(t) = L \frac{di_L(t)}{dt}. \tag{34}\]

Applying (31) and substituting \( \tau = \frac{2}{3} \) we have
For $t < 0$ the circuit is in steady state and $v_L = 0$. The plot of $v_L$ against $t$ shows that at $t = 0$ the voltage $v_L$ jumps from 0 to 4 (see Fig. 25).

Alternatively, the voltage $v(t)$, for $t > 0$ can be found applying KVL in the circuit depicted in Fig. 17

$$v_L(t) + (3+3)i(t) - 20 = 0$$

and substituting (33). As a result we obtain

$$v_L(t) = 20 - \frac{10}{3} - \frac{2}{3}e^{\frac{-t}{\tau}} = 4e^{\frac{-t}{\tau}}, \quad t > 0.$$  \hspace{1cm} (37)

**Question 6**

The switch was in position A for a long time prior to $t = 0$. Find the voltage $v_C(t)$ for $t \geq 0$. 

$$v_L(t) = 2\left(\frac{4}{3}e^{\frac{-t}{\tau}}\right) = 4e^{\frac{-t}{\tau}}. \hspace{1cm} (35)$$
Solution
Since the switch was in position A for a long time the circuit is in steady state at $t = 0^-$. Hence, the voltage $v_C$ is constant and the current $i_C$, given by

$$i_C = C \frac{dv_C}{dt}$$

is equal to zero. As a result the capacitor can be removed and the circuit at $t = 0^-$ is as shown in Fig. 27.

![Fig. 27](image)

In this circuit

$$i_1(0^-) = i_3(0^-) = \frac{E}{R_1 + R_3} = 1\text{A}$$

and

$$v_C(0^-) = -R_{3d}(0^-) = -10\text{V} = v_C(0).$$

For $t > 0$ the circuit is as shown in Fig. 28.

![Fig. 28](image)

As $t \to \infty$, the voltage $v_C$ is constant and $i_C(\infty) = 0$. Hence, the capacitor can be removed and the circuit becomes as depicted in Fig. 29.
In this circuit we write

\[ i_1(\infty) = i_2(\infty) = \frac{E}{R_1 + R_2} = 1 \text{ A} \ , \quad (41) \]

\[ v_C(\infty) = R_2 i_2(\infty) = 10 \text{ V} \ . \quad (42) \]

To find \( v_C(t) \) we substitute (40) and (42) into the equation

\[ v_C(t) = v_C(\infty) + (v_C(0) - v_C(\infty))e^{\frac{-t}{\tau}} , \quad (43) \]

where

\[ \tau = R_{eq}C \ . \quad (44) \]

To find \( R_{eq} \) we consider the circuit shown in Fig. 28. In this circuit we set \( E = 0 \), remove the capacitor \( C \) and find the resistance of the one-port KL

\[ R_{eq} = R_3 + \frac{R_1 R_2}{R_1 + R_3} = 17.5 \Omega \ . \quad (45) \]

Thus, it holds \( \tau = 17.5 \cdot 10^{-6} \text{ s} \) and

\[ v_C(t) = 10 + (-10 - 10)e^{\frac{-t}{\tau}} = \left( 10 - 20e^{\frac{-t}{\tau}} \right) \text{ V} \ . \quad (46) \]

Plot of \( v_C(t) \) is shown in Fig. 30.
Fig. 30
AC analysis

Question 1

Fig. 1

(i) In the circuit, depicted in Fig. 1, find the sinusoidal voltage $v_S(t)$ knowing that

$$v_L(t) = (10 \cos(500t + 30^\circ)) \, V.$$ 

(ii) Compute the reactive powers entering the inductor $L$ and the capacitor $C$ as well as the average power consumed by the resistor $R_2$.

(iii) Sketch a phasor diagram.

Data: $R_1 = R_2 = 10 \, \Omega$, $L = 0.01 \, \text{H}$, $C = 0.2 \, \text{mF}$.

Solution

(i) We redraw the circuit using the phasor format as shown in Fig. 2.

Fig. 2

The phasor of the voltage $v_L(t)$ is

$$V_L = 10e^{j30^\circ} \, V.$$
To determine the voltage source we proceed as follows. At first we compute the current $I_3$

$$I_3 = \frac{V_L}{Z_L},$$

where

$$Z_L = j\omega L = j5\Omega = 5e^{j90}\Omega.$$

Hence, it holds

$$I_3 = \frac{10e^{j30}}{5e^{j90}} = 2e^{j60} = 2(\cos 60^\circ - j\sin 60^\circ) = (1 - j1.732) A.$$

Then we compute the voltage $V_2$

$$V_2 = V_L + V_C = I_3(Z_L + Z_C),$$

where

$$Z_C = -j\frac{1}{\omega C} = -j10\Omega = 10e^{-j90}\Omega.$$

According to the foregoing

$$V_2 = 2e^{j60} (j5-j10) = 10e^{j150} = 10 (\cos 150^\circ - j \sin 150^\circ) = (-8.66 - j5)V.$$

Using Ohm’s law we find the current $I_2$

$$I_2 = \frac{V_2}{R_2} = e^{-j150} = (-0.866 - j0.5) A.$$

KCL applied to the bottom node leads to the equation

$$-I_2 + I_1 - I_3 = 0$$

or

$$I_1 = I_2 + I_3 = (0.134 - j2.232) A.$$

The results obtained above enable us to determine the source voltage

$$V_S = V_2 + R_1I_1 = (-7.32 - j27.32) V = 28.28 e^{j255} V.$$

Now we go from the phasor $V_S$ to the time varying voltage $v_S(t)$

$$v_S(t) = \Re(V_S e^{j\omega t}) = 28.28 \cos(500t + 255^\circ) = 28.28 \cos(500t - 105^\circ) V.$$

(ii) The reactive powers entering the inductor $L$ and the capacitor $C$ are as follows:

$$P_{X_L} = \frac{1}{2} |I_S|^2 X_L = 10 \text{ VAR},$$
$$P_{xc} = \frac{1}{2} |I_3|^2 X_C = -20 \text{ VAR},$$

whereas the average power consumed by the resistor $R_2$ is

$$P_{av_2} = \frac{1}{2} R_2 |I_2|^2 = 5 \text{ W}.$$ 

(iii) A phasor diagram is sketched in Fig. 3.

![Fig. 3](image)

**Question 2**

(i) Compute the resonant angular frequency $\omega_0$ of the parallel combination $L_1$ and $C_1$.

(ii) Determine the branch currents $i_2(t), i_1(t), i_{L_1}(t), i_{C_1}(t)$ if

$$v_s(t) = 60 \cos(\omega_0 t + 60^\circ) \text{ V}.$$ 

(iii) Sketch a phasor diagram.

Data: $R_1 = 100 \Omega$, $R_2 = 20 \Omega$, $L_1 = 0.02 \text{ H}$, $L_2 = 0.01 \text{ H}$, $C_1 = 0.5 \mu\text{F}$. 

![Fig. 4](image)
Solution

(i) The resonant angular frequency is given by

\[ \omega_0 = \frac{1}{\sqrt{L_1 C_1}} = \frac{1}{\sqrt{2 \cdot 10^{-2} \cdot 0.5 \cdot 10^{-6}}} = 10^4 \text{ rad/s}. \]

(ii) We redraw the circuit in the phasor format (see Fig. 5).

![Fig. 5](image-url)

Having \( v_s(t) \) we find the voltage source phasor

\[ V_s = 60^{60^\circ} \text{V}. \]

Since at resonance \( I = 0 \), the same current \( I_1 = I_2 \) traverses \( R_1, Z_{L_2}, R_2 \) and \( V_s \). Hence, we have

\[ I_2 = I_1 = \frac{V_s}{R_2 + Z_{L_2} + R_1} = \frac{60e^{60^\circ}}{120 + j100} = 0.384e^{120.2^\circ} \text{A} \]

and in the time domain

\[ i_2(t) = i_1(t) = \text{Re}(I_2e^{j\omega t}) = 0.384 \cos(10^4 t + 20.2^\circ) \text{A}. \]

To determine the current \( I_{L_1} \) we find the voltage \( V_1 \)

\[ V_1 = R_1 I_1 = 38.4e^{120.2^\circ} \text{V} \]

and apply Ohm’s law

\[ I_{L_1} = \frac{V_1}{j\omega L_1} = \frac{38.4e^{120.2^\circ}}{200e^{j90^\circ}} = 0.192e^{-j69.8^\circ} \text{A}. \]

Hence, we obtain

\[ i_{L_1}(t) = \text{Re}(I_{L_1}e^{j\omega t}) = 0.192 \cos(10^4 t - 69.8^\circ) \text{A}. \]

Since at resonance \( i_{L_1}(t) = -i_{L_2}(t) \) then
\[ i_c(t) = 0.192 \cos (10^4 t - 69.8^\circ + 180^\circ) = 192 \cos (10^4 t + 110.2^\circ) \text{A} \].

(i) A phasor diagram is sketched in Fig. 6

![Fig. 6](image)

**Question 3**
In the circuit shown in Fig. 7 find the sinusoidal voltage \( v_c(t) \) using the phasor concept. Compute the average and reactive power entering the branch consisting of \( R_2 \) and \( L \). Sketch a phasor diagram.

![Fig. 7](image)

**Data:**
\[ v_s(t) = 20 \cos (500t + 30^\circ) \text{V}, \quad R_1 = R_2 = 20 \Omega, \quad L = 0.02 \text{ H}, \quad C = 4 \times 10^{-5} \text{ F} \].
**Solution**

We redraw the circuit of Fig. 3 in the phasor format as depicted in Fig. 8.

![Circuit Diagram](image)

The impedances $Z_c$ and $Z_2$ are:

$$Z_c = -j\frac{1}{\omega C} = -j\frac{1}{500 \cdot 4 \cdot 10^{-5}} = -j50\Omega,$$

$$Z_2 = R_2 + j\omega L = 20 + j500\cdot0.02 = (20 + j10)\Omega.$$

The total impedance faced by the voltage source $V_S$ is

$$Z = R_1 + \frac{Z_cZ_2}{Z_c + Z_2} = 45\Omega.$$  

Since

$$V_S = 20e^{j30^\circ} \text{ V},$$

then

$$I_1 = \frac{V_S}{Z} = 0.44e^{j30^\circ} \text{ A}.$$  

Voltage $V_c$ appears between nodes A and B, hence, it holds

$$V_c = \frac{Z_cZ_2}{Z_c + Z_2}I_1 = 11.1e^{j30^\circ} \text{ V}$$

and in time domain

$$v_c(t) = \text{Re}(V_ce^{j\omega t}) = 11.1\cos(500t + 30^\circ)\text{ V}.$$  

Next we find

$$I_2 = \frac{V_c}{Z_2} = \frac{11.1e^{j30^\circ}}{20 + j10} = 0.49e^{j3.5^\circ} \text{ A}.$$
Observe that the voltage across the branch $R_2$, $L$ is the same as $V_C$. Hence we have

$$P_{av} = \frac{1}{2} |I_2|^2 R_2 = 2.4 \text{ W},$$

$$P_X = \frac{1}{2} |I_2|^2 \omega L = 1.2 \text{ VAR}.$$  

A phasor diagram is sketched in Fig. 9.

**Question 4**

The circuit depicted in Fig. 10 is supplied with the sinusoidal voltage source

$$v_S(t) = 12 \cos (\omega_0 t + 45^\circ) \text{V}, \quad \omega_0 = \frac{1}{\sqrt{L_1 C_1}}.$$

Find the sinusoidal current $i_2(t)$ and the reactive power of the capacitor $C_2$. Sketch a phasor diagram.

Data:

$$R_1 = R_2 = 50 \Omega, \quad L_1 = 0.01 \text{H}, \quad C_1 = 1 \text{µF}, \quad C_2 = 4 \text{µF}.$$
Solution
We consider the circuit shown in Fig. 11.

![Fig. 11](image)

The angular frequency $\omega_0$ is

$$\omega_0 = \frac{1}{\sqrt{L_1 C_1}} = 10^4 \text{ rad/s}. $$

Since $\omega_0$ is the resonant frequency of the series connection of $R_1$, $L_1$, $C_1$ then the impedance of this connection is

$$Z_{R_1L_1C_1} = R_1 = 50 \Omega. $$

Hence, the total impedance faced by the voltage source is given by

$$Z = R_2 + \frac{R_1 Z_c}{R_1 + Z_c}, $$

where

$$Z_c = -j \frac{1}{\omega_0 C_2} = -25 \Omega. $$

Thus, we have

$$Z = 50 + \frac{50(-j25)}{50 - j25} = 60 - j20 = 63.2e^{-j18.4^\circ} \Omega$$

and

$$I_2 = \frac{V_s}{Z} = \frac{12e^{j45^\circ}}{63.2e^{-j18.4^\circ}} A = 019e^{j63.4^\circ} A, $$

$$i_2(t) = \text{Re}(I_2 e^{j\omega_0 t}) = 0.19 \cos(10^4 t + 63.4^\circ) A.$$
Next we compute voltage $V_{c_2}$ using KVL:

$$V_{c_2} = V_s - R_2I_2 = 4.23V .$$

To find the reactive power of the capacitor $C_2$ we use the relationship

$$P_x = -\frac{1}{2}B|V_{c_2}|^2,$$

where

$$B = \omega_0 C_2 = 0.04S .$$

Hence, we obtain

$$P_x = -0.36VAR .$$

A phasor diagram is sketched in Fig. 12.
**Question 5**
In the circuit depicted in Fig. 13 determine the sinusoidal source voltage \( v_S(t) \) if
\[
i_2(t) = 1.5 \cos(1000t + 60^\circ) \text{V}.
\]

Compute the reactive power entering the one-port AB. Sketch a phasor diagram.

![Fig. 13](image)

Data:
\( R_1 = R_2 = 10\Omega, \quad L = 0.02\text{H}, \quad C = 2 \cdot 10^{-5} \text{F}. \)

**Solution**
Let us redraw the circuit using the phasor format as depicted in Fig. 14, where
\[
Z_{AB} = \frac{j\omega L \left( -j \frac{1}{\omega C} \right)}{j\omega L - j \frac{1}{\omega C}} = \frac{j20(-j50)}{-j30} = j33.3\Omega.
\]

![Fig. 14](image)
The phasor of the current traversing all the elements of the circuit shown in Fig. 14 is

\[ I_2 = 1.5e^{j60^\circ} \text{ A}. \]

To find \( V_s \) we compute the total impedance faced by the voltage source

\[ Z = R_1 + Z_{AB} + R_2 = (20 + j33.3) \Omega \]

and expressed it in polar from

\[ Z = \sqrt{20^2 + (33.3)^2} e^{j \tan^{-1} \frac{33.3}{20}} = 38.9e^{j59^\circ} \Omega. \]

Ohm’s law in phasor from gives

\[ V_s = ZI_2 = 58.3e^{j19^\circ} \text{ V}. \]

Now we come from the phasor back to the sinusoidal function using the expression

\[ v_s(t) = \text{Re} \left( V_s e^{j\omega t} \right), \]

where \( \omega = 1000 \frac{\text{rad}}{s} \). As a result we obtain

\[ v_s(t) = 58.3 \cos(1000t + 119^\circ) \text{ V}. \]

The reactive power entering the one-port AB is specified by

\[ P_x = \text{Im} \left( \frac{1}{2} Z_{AB} |I_2|^2 \right) = \text{Im} \left( j \frac{1}{2} 33.3 \cdot (1.5)^2 \right) = 37.5 \text{ VAR}. \]

A phasor diagram is shown in Fig. 15.
Question 6
In the circuit shown in Fig. 16 the voltage $v_L(t)$ across the inductor $L$ is given:

$$v_L(t) = 3 \cos(1000t + 60^\circ)V.$$ 

(i) Find the sinusoidal source voltage $v_s(t)$ using the phasor concept.
(ii) Compute the average and reactive power delivered to the branch AB consisting of $L$ and $R_3$.
(iii) Sketch a phasor diagram.

Data: $R_1 = 50\Omega$, $R_2 = 40\Omega$, $R_3 = 60\Omega$, $L = 0.03H$.

Solution
We redraw the circuit of Fig.16 in the phasor form (see Fig. 17).

(i) We find the phasor of the voltage $v_L(t)$

$$V_L = 3e^{j60^\circ}V$$
and the inductor impedance

\[ Z_L = j\omega L = j \cdot 1000 \cdot 0.03 = j30 = 30e^{j0^\circ}\Omega. \]

Then we compute the phasor \( I_L \)

\[ I_L = \frac{V_L}{Z_L} = \frac{3 \cdot e^{j60^\circ}}{30e^{j90^\circ}} = 0.1e^{-j30^\circ} = 0.1(\cos 30^\circ - j\sin 30^\circ) = (0.087 - j0.05)A \]

and find the phasor of the voltage \( v_2(t) \)

\[ V_2 = (R_3 + Z_L)I_L = (60 + j30)(0.087 - j0.05) = (6.72 - j0.39)V. \]

Having \( V_2 \) we compute \( I_2 \)

\[ I_2 = \frac{V_2}{R_2} = \frac{6.72 - j0.39}{40} = (0.17 - j0.01)A. \]

Using KCL at the top node we obtain

\[ I_1 = I_2 + I_L = (0.26 - j0.06)A. \]

KVL applied to the loop consisting of \( V_S, R_1, R_2 \) leads to the equation

\[ V_S = RI + V_2 = 50(0.26 - j0.06) + 6.72 - j0.39 = (19.57 - j3.39)V = \sqrt{19.57^2 + 3.39^2} e^{-j\tan^{-1}\frac{3.39}{19.57}} = 19.86e^{-j9.8^\circ}V. \]

Hence, we have

\[ v_S(t) = \text{Re}(V_se^{j\omega t}) = 19.86\cos(1000t - 9.8^\circ)V. \]

(ii) The complex power of the branch AB is given by

\[ P_{AB} = \frac{1}{2}V_2I_L^* = \frac{1}{2}(6.72 - j0.39)(0.087 + j0.05) = (0.302 + j0.151)VA. \]

Thus, it holds

\[ P_{av} = \text{Re}(P_{AB}) = 0.302 \text{ W}, \]

\[ P_X = \text{Im}(P_{AB}) = 0.151 \text{ VAR}. \]
(i) A phasor diagram is sketched in Fig. 18

![Fig. 18](image)

**Question 7**

In the circuit depicted in Fig. 19 determine the resonant angular frequency $\omega_0$ of the branch AB consisting of $R_1$, $L_1$, $C_1$.

The circuit is driven by the voltage source

$$v_S(t) = 100\cos(\omega_0 t + 45^\circ) \text{ V}.$$ 

(i) Find the capacitance $C_2$ knowing that the current $i$ (flowing through $C_2$) leads voltage $v_S$ by $45^\circ$.

(ii) Compute $i(t)$.

Data: $R_1 = R_2 = 1000\, \Omega$, $L_1 = 0.04\, \text{H}$, $C_1 = 1\, \mu\text{F}$.

![Fig. 19](image)
Solution
First we compute the resonant angular frequency

\[ \omega_0 = \frac{1}{\sqrt{L_1 C_1}} = \frac{1}{\sqrt{0.04 \cdot 10^{-6}}} = 5000 \text{ rad/s} \]

and the voltage source

\[ v_S(t) = 100 \cos(5000t + 45^\circ)V \]

Since the branch AB is in resonance, its impedance equals \( R_1 \). Consequently, the circuit can be redrawn as shown in Fig. 20, where

\[ R = \frac{R_1 R_2}{R_1 + R_2} = 500\Omega. \]

Fig. 20

(i) Current I is given by the expression

\[ I = \frac{V_S}{R - j \frac{1}{\omega_0 C_2}}, \]

where

\[ V_S = 100e^{45^\circ} \text{ V}. \]

Hence, we obtain

\[ R - j \frac{1}{\omega_0 C_2} = \frac{V_S}{I}. \]

Since \( I \) leads \( V_S \) by 45°, its phase is 90°. Thus, it holds

\[ I = I|e^{90^\circ}| \]

and
\[ \frac{V_\phi}{I} = \frac{100}{|I|} e^{-j\theta} . \]

Using the above results we write

\[ \tan^{-1} \left( \frac{-1}{R \omega C_2} \right) = -45^\circ \]

or

\[ \frac{1}{R \omega C_2} = 1 . \]

Solving for \( C_2 \) we obtain

\[ C_2 = \frac{1}{R \omega} = \frac{1}{500 \cdot 5000} = 0.4 \cdot 10^{-6} \text{ F} . \]

(ii) To find the current \( i(t) \) we use equation

\[ I = \frac{100e^{j45^\circ}}{500 - j500} = \frac{100e^{j45^\circ}}{\sqrt{500^2 + 500^2} e^{-j45^\circ}} = 1.414e^{j90^\circ} \text{ A} \]

and go from the frequency domain to the time domain

\[ i(t) = \text{Re}(e^{j300t}) = 1.414 \cos(5000t + 90^\circ) \text{ A} . \]
Three-phase systems

Question 1
In the three-phase system shown in Fig. 1 the terminals B’ and C’ of the load have been short circuited. The system is supplied with a balanced three-phase generator. The amplitude of the generator phase voltage is 100V, the impedance of each connecting wire is \( Z_p = (1 + j) \Omega \) and the phase impedance of the balanced load is \( Z = (6 + j10) \Omega \). Find the line currents \( I_a, I_b, I_c \) and sketch a phasor diagram.

Solution
We redraw the circuit as shown in Fig. 2.

Let
\[
Z_a = Z_p + Z + \frac{1}{2}Z = (10 + j16) \Omega ,
\]
\[
Z_b = Z_p = (1 + j) \Omega ,
\]
\[
Z_c = Z_p = (1 + j) \Omega ,
\]
then the circuit depicted in Fig. 2 can be simplified as shown in Fig. 3.
The generator phase voltages are as follows:

\[ V_a = 100\text{V}, \]
\[ V_b = \left( \frac{1}{2} - j\frac{\sqrt{3}}{2} \right) 100 = (-50 - j86.6)\text{V}, \]
\[ V_c = \left( \frac{1}{2} + j\frac{\sqrt{3}}{2} \right) 100 = (-50 + j86.6)\text{V}. \]

We find the voltage \( V_n \)

\[ V_n = \frac{V_a + V_b + V_c}{\frac{1}{Z_a} + \frac{1}{Z_b} + \frac{1}{Z_c}} = (-44.8 - j1.2)\text{V} \quad (1) \]

and next calculate the currents

\[ I_a = \frac{V_a - V_n}{Z_a} = (4.1 - j6.5)\text{A}, \quad (2) \]
\[ I_b = \frac{V_b - V_n}{Z_b} = (-45.3 - j40.1)\text{A}, \quad (3) \]
\[ I_c = \frac{V_c - V_n}{Z_c} = (41.2 + j46.5)\text{A}. \quad (4) \]

The phasor diagram is shown in Fig. 4.
Question 2
In the three-phase system shown in Fig. 5 the phase c is open circuited. The system is supplied with a balanced three-phase generator. The amplitude of the generator phase voltage is 220V, the impedance of each connecting wire is $Z_p = (3+j 4) \, \Omega$ and the phase impedance of the balanced load is $Z = (27+j 36) \, \Omega$. Find $I_a$, $I_b$, $\tilde{V_o}$ and sketch a phasor diagram.

Solution
Since $I_c = 0$, then $I_b = -I_a$. We write KVL equation in the loop consisting of the phases a and b:

$$V_a - Z_p I_a - Z I_a + Z(-I_a) + Z_p(-I_a) - V_b = 0.$$ 

Hence, we find

$$I_a = \frac{V_a - V_b}{2Z + 2Z_p}. \quad (5)$$

The generator phase voltages are
\[ V_a = 220 \text{ V}, \]
\[ V_b = V_a \left( -\frac{1}{2} - j\frac{\sqrt{3}}{2} \right) = (-110 - j190.5) \text{ V}, \]
\[ V_c = V_a \left( -\frac{1}{2} + j\frac{\sqrt{3}}{2} \right) = (-110 + j190.5) \text{ V}. \]

Substituting these voltages into (5) yields
\[
I_a = \frac{330 + j190.5}{60 + j80} = (3.5 - j1.6) \text{ A} = -I_b.
\]

To find \( \vec{V}_o \) we apply KVL
\[
\vec{V}_o - V_c + V_n = 0, \tag{6}
\]
where
\[
V_n = \frac{V_a}{Z_p + Z} + \frac{V_b}{Z_p + Z} + \frac{V_c}{Z_c}, \tag{7}
\]

In equation (7) \( |Z_c| \to \infty \), hence, we have
\[
V_n = \frac{V_a + V_b}{2} = -\frac{V_c}{2}. \tag{8}
\]

Using (6) we find
\[
\vec{V}_o = V_c - V_n = V_c + \frac{V_c}{2} = \frac{3}{2} V_c = (-165 + j285.75) \text{ V}. \]

The phasor diagram is shown in Fig. 6.
Question 3
In the three-phase system shown in Fig. 7, the phase c of the load is short circuited. The system is supplied with a balanced three-phase generator. The amplitude of the generator phase voltage is 100 V, the impedance of the balanced load \( Z = 100 \Omega \). Find the currents \( I_a, I_b, I_c \) and the voltages \( V_a' \) and \( V_b' \).

![Fig. 7](image)

Solution
Using KVL we write
\[
V_a = V_c. \tag{9}
\]
Hence, we have
\[
V_a' = V_a - V_c, \tag{10}
\]
\[
V_b' = V_b - V_c. \tag{11}
\]
We find the source voltages
\[
V_a = 100V, \tag{12}
\]
\[
V_b = 100\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right) = (-50 - j86.6)V, \tag{13}
\]
\[
V_c = 100\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) = (-50 + j86.6)V \tag{14}
\]
and substitute them into the equations (10) and (11)
\[
V_a' = V_a - V_c = (150 - j86.6)V, \]
\[
V_b' = V_b - V_c = -j173V.
\]
Next we calculate the currents
\[
I_a = \frac{V_a'}{Z} = (1.5 - j0.87)A,
\]
\[
I_b = \frac{V_b'}{Z} = (-j1.73)A,
\]
\[
I_c = -(I_a + I_b) = (-1.5 + j2.6)A.
\]
The phase diagram is shown in Fig. 8.
Question 4

In the three-phase system shown in Fig. 9 determine the indications of the wattmeters and the average power consumed by the load. The system is supplied with a balanced three-phase generator. The amplitude of the generator phase voltage is 324V, the impedances of the load are: $Z_a = 100\,\Omega$, $Z_b = (100 - j100)\,\Omega$, $Z_c = 200\,\Omega$.

Solution

The indications of the wattmeters are

$$P_{w_1} = \frac{1}{2} \Re(V_{ac} I_a^*)$$  \hspace{1cm} (15)

$$P_{w_2} = \frac{1}{2} \Re(V_{bc} I_b^*)$$  \hspace{1cm} (16)
At first we find the phase voltages of the generator and the voltage $V_n$:

\[
V_a = 324 \, \text{V},
\]

\[
V_b = 324 \left( -\frac{1}{2} - j\frac{\sqrt{3}}{2} \right) = (-162 - j280.3) \, \text{V},
\]

\[
V_c = 324 \left( -\frac{1}{2} + j\frac{\sqrt{3}}{2} \right) = (-162 + j280.3) \, \text{V},
\]

\[
V_n = \frac{V_a}{Z_a} + \frac{V_b}{Z_b} + \frac{V_c}{Z_c} = (132.6 - j73.6) \, \text{V}.
\]

Next we compute the currents $I_a$ and $I_b$:

\[
I_a = \frac{V_a - V_n}{Z_a} = \frac{324 - (132.6 - j73.6)}{100} = (1.91 + j0.74) \, \text{A},
\]

\[
I_b = \frac{V_b - V_n}{Z_b} = \frac{-162 - j280.3 - (132.6 - j73.6)}{100 - j100} = (-0.44 - j2.51) \, \text{A}
\]

and apply the equations (15) and (16).

\[
P_{W_1} = \frac{1}{2} \text{Re}((V_a - V_c)(1.91 - j0.74)) = \frac{1}{2} \text{Re}((486 - j280.3)(1.91 - j0.74)) = \frac{1}{2} (486 \cdot 1.91 - 280.3 \cdot 0.74) = 360.4,
\]

\[
P_{W_2} = \frac{1}{2} \text{Re}((V_b - V_c)(-0.44 + j2.51)) = \frac{1}{2} \text{Re}((-j560.6)(-0.44 + j2.51)) = \frac{1}{2} 560.6 \cdot 2.51 = 703.6.
\]

To find the average power consumed by the load we use the formula

\[
P = P_{W_1} + P_{W_2} = 1064 \, \text{W} = 1.064 \, \text{kW}.
\]