1. Description and Properties of Signals and Systems

1.1. Introduction

Generally, a signal is a function of one or more independent variables. However, we will consider the signals having one independent variable only; furthermore, this variable will be restricted to time.

Examples of signals include: time depending voltages and currents in an electric circuit, the variation in a gross national product, music waveforms, the variation of atmospheric temperature.

If a signal is represented at all instants of time, it is said to be a continuoustime signal or simply a continuous signal. A signal which is specified at discrete instants of time is said to be a discrete-time signal or simply a discrete signal. Discrete signals occur either due to the nature of the process, e.g. the variation in the number of cars crossing the border every day, or due to the sampling process. Examples of continuous and discrete signals are shown in Fig.1.1.



Fig. 1.1. Examples of continuous and discrete signals

For the discrete signal, time can only take discrete values; therefore we write $x(nT_s)$, where T_s is the time between samples and n is the number of the sample. However, not all discrete signals are formed by the sampling of the continuous signal and in such a case the signal is usually written x(n). Also with sampled signals, in general, we will omit the T_s and write x(n). Sometimes the sampling interval is not constant and changes from one step to another, but such a case will not be considered in this book.

1.2. Some properties of signals

In this section we will discuss some properties of both continuous and discrete signals.

Reflection

Let us consider a signal x(t); the reflected signal is described by x(-t). Thus, the reflected signal assumes at time -t the value of the original signal that occurs at time t. This is illustrated in Fig.1.2.



Fig. 1.2. The reflection operation of continuous signal x(t)

We define reflection for discrete signals similarly (see Fig.1.3).



Fig. 1.3. The reflection operation of discrete signal x(n)

Shifting

Let x(t) be an original signal. To obtain the shifted signal, for a shift t_0 , the value of the original signal that took place at *t* must now occur at $t + t_0$. If t_0 is positive, the shifted signal is called a delayed signal (see Fig.1.4)



Fig. 1.4. The shifting operation of continuous signal x(t)

The shifted signal is specified by $x(t - t_0)$. The shifting property can be directly applied to discrete signals. This is illustrated in Fig.1.5.



Fig. 1.5. The shifting operation of discrete signal x(n)

Periodicity

A continuous signal x(t) is said to be periodic if there exists such a time interval *T* that

$$x(t+T) = x(t) \qquad \text{for all } t . \qquad (1.1)$$

The smallest time T is known as the period. It should be noted that if a signal is periodic with the period T, it is also periodic for any integer multiple of T.

Similarly, a discrete signal x(n) is said to be periodic if there exists such a number N that

$$x(n+N) = x(n) \qquad \text{for all } n \qquad (1.2)$$

The smallest number N is known as the period. Examples of continuous and discrete periodic signals are shown in Fig.1.6.



Fig. 1.6. Examples of periodic signals

1.3. Sinusoidal and exponential signals

1.3.1. Sinusoidal signal

The most important periodic signal is the sinusoidal signal. It is justified since the voltages generated by alternators in power systems have a sinusoidal waveform, a sinusoid has convenient mathematical properties, a periodic signal can be expressed as a sum of sinusoidal terms. The sine and the cosine signals can be represented as follows:

$$x(t) = A\sin\omega t$$
 $y(t) = A\cos\omega t$

where A is the amplitude and ω is the angular frequency. The period T corresponds to angle 2π , hence, the equation

$$\omega T = 2\pi$$

or

$$\omega = \frac{2\pi}{T} = 2\pi f$$

holds, where f is the frequency. Although there is a difference between ω and f, usually we omit the adjective angular and call ω frequency.

A general sinusoidal signal has the form

$$x(t) = A\cos(\omega t + \alpha)$$
(1.3)

where α is known as the phase.

A discrete sinusoidal signal can be described by the relationship

$$x(n) = A\cos(n\omega T_s + \alpha).$$
(1.4)

This signal is obtained by sampling the continuous signal

$$x(t) = A\cos(\omega t + \alpha)$$

with the sampling interval T_s . Since the sinusoid is a periodic function then

$$x(n) = A\cos(n\omega T_{s} + \alpha) = A\cos(n(\omega T_{s} + 2\pi k) + \alpha) =$$
$$= A\cos\left(n\left(\omega + \frac{2\pi k}{T_{s}}\right)T_{s} + \alpha\right)$$
(1.5)

where k is an integer. The discrete sinusoid on the right hand side of (1.5) can be considered as the sampled continuous sinusoid with the sampling interval T_s and angular frequency

$$\omega + \frac{2\pi k}{T_s} = \omega + k\omega_s = 2\pi (f + kf_s)$$

where f_s is the sampling frequency. Thus, the discrete signal (1.4), sampling a sinusoid with angular frequency ω , will also be a sampling signal of any sinusoid with angular frequency $\omega + k\omega_s$ where k is an integer. Hence, there are infinitely many continuous sinusoidal signals corresponding to the discrete signal (1.4). This leads to the conclusion that having the sampled signal (1.4) it is not possible to determine which of the continuous sinusoids is represented by these samples. Thus, there is an ambiguity between the sinusoid with the frequency ω and the sinusoids with the frequencies $\omega + k\omega_s$. Consequently, the possibility of identifying the original signal by examining the sampled signal is lost. This effect is known as aliasing.

To illustrate the aliasing effect, we consider a sinusoidal signal having the frequency f = 100Hz. This signal is sampled with the frequency $f_s = 700$ Hz. Using these samples, an ambiguity arises between this signal and a signal of

frequency $f + kf_s = 100 + k700$, where k is an arbitrary integer. The aliasing effect for k = 1 $(f + kf_s = 800 \text{Hz})$ is illustrated in Fig.1.7.



Fig. 1.7. Illustration of the aliasing effect

1.3.2. Exponential signal

Let us consider an exponential signal of the form

$$x(t) = A e^{j(\omega t + \alpha)}.$$
 (1.6)

Using Euler's identity we obtain

$$x(t) = A\cos(\omega t + \alpha) + jA\sin(\omega t + \alpha).$$
(1.7)

Thus, the equations

$$A\cos(\omega t + \alpha) = \operatorname{Re}(Ae^{j(\omega t + \alpha)})$$
(1.8)

and

$$A\sin(\omega t + \alpha) = Im(Ae^{j(\omega t + \alpha)})$$
(1.9)

hold.

Thus, sinusoidal signals can be expressed in terms of the complex exponential signal.

A discrete exponential signal has the form

$$x(n) = A e^{j(n\omega T_S + \alpha)}.$$
 (1.10)

Applying the Euler expression we find

$$x(n) = A\cos(n\omega T_s + \alpha) + jA\sin(n\omega T_s + \alpha).$$
(1.11)

Equation (1.11) implies that the discrete sinusoidal signals can be expressed in terms of the discrete complex exponential as follows:

$$A\cos(n\omega T_{S} + \alpha) = \operatorname{Re}\left(Ae^{j(n\omega T_{S} + \alpha)}\right)$$
(1.12)

$$A\sin(n\omega T_{S} + \alpha) = Im\left(Ae^{j(n\omega T_{S} + \alpha)}\right).$$
(1.13)

1.4. The unit step and the unit impulse

Our objective in this section is to define and analyze commonly used signals: the unit step and the unit impulse in the continuous time and the unit step sequence and the unit sample sequence in the discrete time. The unit step function is defined as follows:

$$u(t) = 0$$
 for $t < 0$
 $u(t) = 1$ for $t > 0$. (1.14)

At t = 0 a discontinuity occurs. A plot of u(t) is shown in Fig.1.8.



Fig. 1.8. Continuous-time unit step function

To define the unit impulse we consider a rectangular pulse function as depicted in Fig.1.9.



Fig. 1.9. Rectangular pulse function $\Delta_{\varepsilon}(t)$

This function is described by the relationship

$$\Delta_{\varepsilon}(t) = \begin{cases} 0 & \text{if} & t < 0\\ \frac{1}{\varepsilon} & \text{if} & 0 < t < \varepsilon \\ 0 & \text{if} & t > \varepsilon \end{cases}$$
(1.15)

Note that the area under $\Delta_{\varepsilon}(t)$ is

$$\int_{-\infty}^{\infty} \Delta_{\varepsilon}(t) dt = \varepsilon \cdot \frac{1}{\varepsilon} = 1.$$
(1.16)

As ε decreases, the width of the rectangle decreases, the height increases in such a manner that the area remains the same (see Fig.1.10).



Fig. 1.10. Rectangular pulse function $\Delta_{\varepsilon}(t)$ for ε smaller than in Fig. 1.9

As $\varepsilon \to 0$, the pulse function becomes the unit impulse or the Dirac delta function

$$\delta(t) = \begin{cases} 0 & \text{for } t \neq 0\\ \text{singular} & \text{at } t = 0 \end{cases}$$
(1.17)

with

$$\int_{-a}^{a} \delta(t) dt = 1$$
(1.18)

for any real a > 0.

Thus, the following relation holds

$$\int_{-\infty}^{t} \delta(\tau) \mathrm{d} \tau = \begin{cases} 1 & \text{for} & t > 0 \\ 0 & \text{for} & t < 0 \end{cases} = \mathrm{u}(t).$$

We know from the mathematical analysis that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a}^{t}g(\tau)\mathrm{d}\,\tau=g(t).$$

Applying this rule to the integral that defines u(t) above, we obtain

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^{t} \delta(\tau) \mathrm{d}\tau = \delta(t).$$
(1.19)

A comment is important at this stage. We do not have any formal rule, on the ground of mathematical analysis, of deriving the equation in this way because the unit impulse is not a function in the classical sense. The unit step has a discontinuity point at the origin and the classical derivative does not exist. So we have derived this equation in an intuitive way. The validity of this operation can be strictly proved using the distribution theory, which is a branch of mathematics.

An extremely important feature of the unit impulse is its behavior as a combination with another function, as below

$$\int_{-\infty}^{\infty} f(t) \delta(t-t_0) \mathrm{d}t$$

where t_0 is any positive or negative real number.

Since $\delta(t-t_0) = 0$ for all $t \neq t_0$ then

$$f(t)\delta(t-t_0) = f(t_0)\delta(t-t_0).$$

Furthermore, relationship (1.18) implies

$$\int_{-\infty}^{\infty} \delta(t-t_0) = 1.$$

Using the above equations we obtain

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt = f(t_0).$$
(1.20)

The counterparts of the unit step and the unit impulse in the discrete time are the unit step sequence called also the discrete-time unit step function and the unit sample sequence called also the unit pulse function or the Kronecker delta function, respectively. The unit step sequence u(n) is defined as follows:

$$u(n) = 0 \quad \text{for} \quad n < 0$$

$$u(n) = 1 \quad \text{for} \quad n \ge 0.$$
 (1.21)

This is illustrated in Fig.1.11.



Fig. 1.11. Unit-step sequence u(n)

The unit sample sequence $\delta(n)$ is defined by the relationship:

$$\delta(n) = 0 \quad \text{for} \quad n \neq 0$$

$$\delta(n) = 1 \quad \text{for} \quad n = 0.$$
(1.22)

This is illustrated in Fig.1.12.



Fig. 1.12. Unit sample $\delta(n)$

The unit sample can be expressed in terms of the unit step

$$\delta(n) = u(n) - u(n-1) \tag{1.23}$$

and conversely the unit step can be expressed in terms of the unit sample

$$u(n) = \sum_{m=-\infty}^{n} \delta(m).$$
(1.24)

1.5. Continuous and discrete signal representation

1.5.1. Continuous signal representation

Consider the continuous signal x(t) as shown by a smooth line in Fig.1.13.



Fig. 1.13. Signal x(t) and its approximation

This signal can be approximated, in the time interval $[t_0, \infty)$ as a staircase function consisting of rectangles with heights $x(t_k)$ and identical width ε , where

$$\varepsilon = t_{k+1} - t_k \qquad \qquad k = 0, 1, 2, \cdots$$

Using the rectangular pulse function, depicted in Fig.1.9 and specified by equation (1.15), we describe *k*-th rectangle by expression

$$x(t_k) \mathcal{E} \Delta_{\mathcal{E}}(t-t_k)$$

where $\varepsilon \Delta_{\varepsilon}(t - t_k)$ is a shifted by t_k function $\varepsilon \Delta_{\varepsilon}(t)$ (see Fig.1.14).



Fig. 1.14. Signal $\mathcal{E}\Delta_{\mathcal{E}}(t)$ shifted by t_k

Hence, the step approximation of the function x(t) is

$$\sum_{k=0}^{\infty} x(t_k) \mathcal{E} \Delta_{\mathcal{E}}(t-t_k).$$
(1.25)

If $\varepsilon \to 0$ the step approximation becomes the actual signal x(t), $\Delta_{\varepsilon}(t) \to \delta(t)$, and the sum (1.25) becomes an integral as follows

$$x(t) = \int_{t_0}^{\infty} x(\tau) \delta(t-\tau) \mathrm{d}\tau \qquad t > t_0 . \qquad (1.26)$$

Letting $t_0 \rightarrow -\infty$ we have

$$x(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t-\tau) d\tau. \qquad (1.27)$$

If we consider the signal x(t) for a positive t only, then

$$x(t) = \int_{0}^{\infty} x(\tau) \delta(t-\tau) \mathrm{d}\tau . \qquad (1.28)$$

Furthermore, the upper limit of integration can be replaced by t because $\delta(t-\tau) = 0$ for $\tau > t$

$$x(t) = \int_{0}^{t} x(\tau) \delta(t-\tau) \mathrm{d}\tau \,. \tag{1.29}$$

Expression on the right hand side of (1.27) is known as the convolution integral formula. The convolution will be discussed in detail in Section 1.8.

1.5.2. Discrete signal representation

Let us consider an example of a discrete signal shown in Fig.1.15. This signal can be represented by the weighted sum of shifted unit samples

Fig. 1.15. An example of a discrete signal

Expression (1.30) can be rewritten in the form

$$x(n) = x(-1)\delta(n+1) + x(0)\delta(n) + x(1)\delta(n-1) + x(2)\delta(n-2) =$$

= $\sum_{k=-1}^{2} x(k)\delta(n-k).$

Generalizing this result we obtain the following expression for any discrete signal

$$x(n) = \sum_{k=-\infty}^{\infty} x(k) \delta(n-k).$$
(1.31)

If we consider the signal x(n) for a positive n only, then

$$x(n) = \sum_{k=0}^{\infty} x(k) \delta(n-k)$$

Furthermore, the upper limit of summation can be replaced by *n* because $\delta(n-k)=0$ for k > n.

The expression on the right hand side of (1.31) is known as the convolution summation formula.

1.6. Classification of systems

1.6.1. Introduction

A system is a mathematical mapping that transforms the input signal into the output signal. Thus, a system is a process in which one signal is transformed into another signal. Usually, the physical system is made as an interconnection of some components.

A system is called continuous-time if both input signal and output signal are continuous-time signals (see Fig.1.16a). A system is called discrete-time if both input signal and output signal are discrete-time signals (see Fig.1.16b).



Fig. 1.16. Examples of continuous-time and discrete-time systems

The relationship between the input and output signals can be expressed in terms of a function, as a set of equations or as a set of all possible inputs and outputs summarized in a table.

For example a continuous-time system can be specified by the formula

$$y(t) = k \int_{0}^{t} x(\tau) \mathrm{d}\tau$$

and a discrete-time system can be specified by the formula

$$y(n) = \frac{x(n) + x(n-1)}{2}.$$

1.6.2. System properties

In this section we will study some fundamental properties of both continuoustime and discrete-time systems.

Additivity

A system is said to be additive if the response due to a sum of inputs is equal to the sum of the responses due to each of the inputs acting alone, i.e.:

$$f(x_1(t) + x_2(t)) = f(x_1(t)) + f(x_2(t))$$

or

$$f(x_1(n) + x_2(n)) = f(x_1(n)) + f(x_2(n)).$$

Homogeneity

A system is said to be homogenous if multiplying the input by a constant results in multiplying the output by the same constant, i.e.:

$$f(kx(t)) = kf(x(t))$$

or

$$f(kx(n)) = kf(x(n)).$$

Linearity

A system is said to be linear if it is both additive and homogeneous, i.e.:

$$f(k_1x_1(t) + k_2x_2(t)) = k_1f_1(x_1(t)) + k_2f_2(x_2(t))$$

or

$$f(k_1x_1(n) + k_2x_2(n)) = k_1f_1(x_1(n)) + k_2f_2(x_2(n))$$

A system which is not linear is said to be nonlinear.

Example 1.1

Let us consider the continuous-time system described by the equation

$$y(t) = c \int_{0}^{t} x(\tau) \mathrm{d}\tau$$

where c is a constant. Let $y_1(t)$ and $y_2(t)$ be the responses of the system due to the input $x_1(t)$ and $x_2(t)$, respectively, that is:

$$y_1(t) = c \int_0^t x_1(\tau) d\tau$$
$$y_2(t) = c \int_0^t x_2(\tau) d\tau.$$

The response of this system due to the input $x(t) = k_1 x_1(t) + k_2 x_2(t)$ is

$$y(t) = c \int_{0}^{t} x(\tau) d\tau = c \int_{0}^{t} (k_1 x_1(\tau) + k_2 x_2(\tau)) d\tau =$$

= $k_1 c \int_{0}^{t} x_1(\tau) d\tau + k_2 c \int_{0}^{t} x_2(\tau) d\tau = k_1 y_1(t) + k_2 y_2(t).$

According to the foregoing relationship the system is linear.

Example 1.2

Consider the discrete-time system specified by the equation

$$y(n)=3x(n)-5x(n-2).$$

The responses of this system to the inputs $x_1(n)$ and $x_2(n)$ are:

$$y_1(n) = 3x_1(n) - 5x_1(n-2)$$
$$y_2(n) = 3x_2(n) - 5x_2(n-2)$$

The response of the system to the input $x(n) = k_1 x_1(n) + k_2 x_2(n)$ is

$$y(n) = 3(k_1x_1(n) + k_2x_2(n)) - 5(k_1x_1(n-2) + k_2x_2(n-2)) = k_1(3x_1(n) - 5x_1(n-2)) + k_2(3x_2(n) - 5x_2(n-2)) = k_1y_1(n) + k_2y_2(n).$$

Thus, the system is linear.

Example 1.3

Consider a continuous-time system described by the equation

$$y(t) = 0.5(x(t))^2.$$

Let $y_1(t)$ and $y_2(t)$ be the responses to the inputs $x_1(t)$ and $x_2(t)$, respectively, i.e.:

$$y_1(t) = 0.5(x_1(t))^2$$

 $y_2(t) = 0.5(x_2(t))^2$.

The response y(t) of this system due to the input

$$x(t) = k_1 x_1(t) + k_2 x_2(t)$$

is

$$y(t) = 0.5(k_1x_1(t) + k_2x_2(t))^2 = 0.5k_1^2(x_1(t))^2 + 0.5k_2^2(x_2(t))^2 + k_1k_2x_1(t)x_2(t).$$

Since the expression on the right hand side is different than

$$k_1 y_1(t) + k_2 y_2(t) = k_1 0.5 (x_1(t))^2 + k_2 0.5 (x_2(t))^2$$

we conclude that the system is nonlinear.

Example 1.4

The system shown in Fig.1.17, including an ideal operational amplifier, is specified by the function $v_{out} = f(v_{in})$ represented by the plot shown in Fig.1.18.



Fig. 1.17. System for Example 1.4



Fig. 1.18. Representation of the system shown in Fig. 1.17

Note that the response of this system to $v_{in} > E_{sat}\beta$ is $v_{out} = E_{sat}$. Hence, for $v_{in} > E_{sat}\beta$ and k > 1 the response of the system to the input kv_{in} is $v_{out} = E_{sat} \neq kE_{sat}$.

For this reason the system is not homogeneous and, the more so it is not linear. However, if v_{in} belongs to the interval $\left[-E_{sat}\beta, E_{sat}\beta\right]$, then the system is described by equation

$$v_{out} = \frac{1}{\beta} v_{in}$$
.

In such a case the response of the system to the input kv_{in} will be kv_{out} as long as $-E_{sat}\beta < kv_{in} < E_{sat}\beta$. Therefore, under this restriction the system can be considered as homogeneous. Furthermore, for the signals $(v_{in})_1$ and $(v_{in})_2$ such that $(v_{in})_1$, $(v_{in})_2$, and $(v_{in})_1 + (v_{in})_2$ belong to the interval $[-E_{sat}\beta, E_{sat}\beta]$ the system is additive. According to the foregoing discussion, for a restricted range of input, the system can be considered as linear.

Time invariance

Let y(t) be the response of a continuous-time system to an input x(t). The system is said to be time-invariant if an input signal x(t-h) causes an output y(t-h) for all t and arbitrary h. This property states that a shift in time of an input signal results in the same time shift in the output signal.

A system which is linear and time-invariant is known as a linear timeinvariant or LTI system. A system which is not time-invariant is said to be timevarying.

Example 1.5

Let us consider the system specified by the equation

$$y(t)=2x(t)-x(t-3).$$

The response of this system to the input x(t-h) is

$$2x(t-h)-x(t-h-3)$$

and equals y(t-h). Thus, the system is time-invariant.

Example 1.6

The system described by the equation

$$y(t) = 3x(t)\cos(\omega_0 t + \alpha)$$

is not time invariant because generally

$$3x(t-h)\cos(\omega_0 t+\alpha) \neq 3x(t-h)\cos(\omega_0 (t-h)+\alpha).$$

A similar property, called shift-invariance (or time-invariance), can be formulated for discrete-time systems.

Let y(n) be the response of a discrete-time system due to an input x(n). The system is said to be shift-invariant (or time-invariant) if an input signal x(n-N) causes an output y(n-N) for all n and arbitrary integer N. A discrete-time system which is both linear and shift-invariant is known as a linear shift-invariant (LSI) or a linear time-invariant (LTI) system.

Example 1.7

Consider the discrete-time system specified by the equation

$$y(n) = x(n) + 2(x(n))^{2} + 5(x(n))^{3}$$
.

The response of this system due to the input x(n-N) is

$$x(n-N)+2(x(n-N))^{2}+5(x(n-N))^{3}$$

and is equal to y(n-N). Thus, the system is time-invariant. However, this system is nonlinear and consequently it is not an LTI system.

Instantaneousness

A continuous-time system is said to be instantaneous if the output in this system at any instant of time depends on the input at that instant only. Otherwise, the system is called non-instantaneous.

Example 1.8

A system specified by the equation

$$y(t) = \int_{0}^{t} x(\tau) \mathrm{d}\tau$$

is non-instantaneous because its response at instant t depends on all time from 0 to t.

Since the responses of non-instantaneous systems depend on previous instants they are said to have a memory. Consequently, instantaneous systems are termed memoryless systems.

Similar properties hold for discrete-time systems. A discrete-time system is said to be instantaneous if the output of this system at any n depends on the input at that n only. Otherwise the system is said to be non-instantaneous.

Example 1.9

The discrete-time system specified by equation

$$y(n) = 2x(n) + x(n-1)$$

is non-instantaneous because its response at n depends on the previous input at n-1.

Example 1.10

The discrete-time system described by the equation

$$y(n) = 5(x(n))^3$$

is instantaneous.

Causality

A continuous-time system is said to be causal if the response of this system, at any instant of time t_0 , depends only on the input up to time t_0 .

A discrete-time system is said to be causal if the response of this system at any n_0 depends on the input up to $n = n_0$.

A general property of causal system is that changes in the output cannot precede changes in the input.

Example 1.11

The system described by the equation

$$y(n) = 2x(n) + 0.5x(n+1)$$

is noncausal because the output at $n = n_0$ depends on the input at $n_0 + 1$. In this book we will study the causal systems only.

Stability

In many applications it is required for a system to produce a bounded output whenever the input is bounded. A system with this property is said to be stable in the bounded input-bounded output sense.

To be technical, we consider a continuous-time system with input x(t) and output y(t). By definition, the system is said to be stable if any bounded input signal x(t), i.e. such that $|x(t)| < K_x$ for all t, produces a bounded output signal y(t), i.e. such that $|y(t)| < K_y$ for all t, where K_x and K_y are positive constants.

The above defined stability is known as bounded-input bounded-output (BIBO) stability.

Example 1.12

Let us consider a system specified by the equation

$$y(t) = \frac{1}{1 - x(t)} \, .$$

Using the bounded input

$$x(t) = \left(1 - e^{-t}\right)u(t)$$

we obtain unbounded output y(t). Thus, the system is not BIBO stable.

Similarly as for continuous systems, we define BIBO stability for discrete-time systems. A discrete system is said to be stable if any bounded input signal x(n), i.e. such that $|x(n)| < K_x$ for all n, produces a bounded output signal y(n), i.e. such that $|y(n)| < K_y$ for all n, where K_x and K_y are positive numbers.

1.7. Response of LTI continuous-time and discrete-time systems

1.7.1. Response of continuous-time LTI systems

Let us consider an LTI system whose response due to the unit impulse $\delta(t)$ is given. The response will be labeled h(t) and termed the unit impulse response. This characterizes the system and plays a very important role in system theory.

Let an input signal x(t) be applied to the system. In Section 1.5 it is shown that the signal x(t) can be approximated by a step function (1.25) repeated below for the reader's convenience

$$\sum_{k=0}^{\infty} x(t_k) \varepsilon \, \Delta_{\varepsilon}(t-t_k). \tag{1.32}$$

Let $h_{\Delta}(t)$ be the response of the system to the rectangular pulse $\Delta_{\varepsilon}(t)$. Since the system is LTI, its response due to the signal (1.32) is

$$\sum_{k=0}^{\infty} x(t_k) \varepsilon h_{\Delta}(t-t_k).$$
(1.33)

Letting $\varepsilon \to 0$ we have: $\Delta_{\varepsilon}(t) \to \delta(t)$, consequently $h_{\Delta}(t) \to h(t)$, and the sum becomes integral. Hence, expression (1.33) becomes the response y(t) to the input x(t) and assumes the form

$$y(t) = \int_{t_0}^{\infty} x(\tau) h(t-\tau) d\tau.$$
 (1.34)

Generally, $t_0 \rightarrow -\infty$ and we may write

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau. \qquad (1.35)$$

We recognize the expression on the right hand side of (1.35) as the convolution integral formula.

If x(t) = 0 for t < 0, then

$$y(t) = \int_{0}^{\infty} x(\tau)h(t-\tau)d\tau. \qquad (1.36)$$

Furthermore, for any causal system $h(t-\tau)=0$ for $t < \tau$ because h(t) is the response to $\delta(t)$ which appears at t = 0 and $h(t - \tau)$, where $t - \tau < 0$, is the response which would precede the input. Hence, we have

$$y(t) = \int_{0}^{t} x(\tau)h(t-\tau)d\tau.$$
 (1.37)

1.7.2. Response of discrete-time LTI systems

Let us consider a discrete-time LTI system in which the response due to the unit sample $\delta(n)$ is given. The response will be labeled h(n) and termed the unit sample response or the unit pulse response. This characterizes the system and plays a very important role in system theory. Section 1.5 shows that the signal x(n) can be expressed in the form (1.31) repeated below

$$x(n) = \sum_{k=-\infty}^{\infty} x(k) \delta(n-k).$$
(1.38)

Since the system is LTI, its response due to $x(k)\delta(n-k)$ is x(k)h(n-k) and the response due to x(n) is

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k).$$
 (1.39)

We recognize expression (1.39) as the convolution summation formula. It constitutes the sum of responses of the system to all the terms of the signal (1.38).

Example 1.13

Let us consider a discrete LTI system whose response to a unit sample is h(n) as shown in Fig.1.19.



Fig. 1.19. Unit sample response of a discrete LTI system

We wish to determine the response of this system to the signal x(n) depicted in Fig.1.20.



Fig. 1.20. Input signal x(n)

The signal x(n) can be represented using expression (1.38)

$$x(n) = \sum_{k=-\infty}^{\infty} x(k) \delta(n-k) = -\delta(n+1) + \delta(n-1) + 2\delta(n-2).$$

Hence, we compute the responses of the system due to the signals $-\delta(n+1)$, $\delta(n-1)$, and $2\delta(n-2)$ (see Fig.1.21).





Fig. 1.21. Response of the system due to signals $-\delta(n+1)$, $\delta(n-1)$, and $2\delta(n-2)$

The response y(n) given by equation

$$y(n) = -h(n+1) + h(n-1) + 2h(n-2)$$

is shown in Fig.1.22.



Fig. 1.22. Response y(n) of the system considered in Example 1.13

The method applied in the above example requires computing the response of the system due to each term of the signal representation (1.38) and summarizing the results. An alternative approach is to consider the graphical interpretation of the convolution summation as it will be explained in Section 1.8.

1.8. Convolution

Section 1.7 shows that the convolution plays a very important role in the analysis and description of both the continuous-time and discrete-time LTI systems. Therefore, in this section we will discuss some properties of the convolution and techniques of performing convolution.

Generally, convolution is a mathematical operation applied to two functions producing a third function. If the two functions are continuous-time functions we refer to convolution as continuous convolution. If they are discrete-time functions we refer to convolution as discrete convolution.

1.8.1. Continuous convolution

Let us consider the functions $f_1(t)$ and $f_2(t)$. The continuous bilateral convolution or simply continuous convolution of $f_1(t)$ and $f_2(t)$ written as

$$f(t) = f_1(t) * f_2(t)$$

is given by

$$f(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) \mathrm{d}\tau \qquad (1.40)$$

where τ is a dummy integration variable. Continuous convolution is commutative, i.e.

$$f_1(t) * f_2(t) = f_2(t) * f_1(t).$$
(1.41)

To prove this property we can use the definition formula

$$f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) \mathrm{d}\tau$$

and perform a change of variables by letting $t - \tau = v$. Then $\tau = t - v$, $d\tau = -dv$, $v \to \infty$ as $\tau \to -\infty$, and $v \to -\infty$ as $\tau \to \infty$. As a result we obtain

$$f_1(t) * f_2(t) = -\int_{\infty}^{-\infty} f_1(t-v) f_2(v) dv = \int_{-\infty}^{\infty} f_2(v) f_1(t-v) dv = f_2(t) * f_1(t)$$

Hence the property.

Applying this property to the result of Section 1.7 we conclude that the response of an LTI system, specified by the unit impulse response h(t), due to an input x(t) can be determined either using equation (1.35) repeated below

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \qquad (1.42)$$

or using the equation

$$y(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau. \qquad (1.43)$$

Thus, for computing y(t) we can apply this expression which leads to simpler integration.

In Section 1.6.2 we defined BIBO stability of continuous-time systems. We now relate this property to the impulse response h(t) of LTI systems with a bounded input x(t), i.e. such that $|x(t)| < K_x$ for all t. Using the convolution equation (1.43) we obtain

$$|y(t)| = \left| \int_{-\infty}^{\infty} h(\tau) x(t-\tau) \mathrm{d}\tau \right| \leq \int_{-\infty}^{\infty} |h(\tau)| |x(t-\tau)| \mathrm{d}\tau \leq K_x \int_{-\infty}^{\infty} |h(\tau)| \mathrm{d}\tau.$$

Hence, if

$$\int_{-\infty}^{\infty} \left| h(\tau) \right| \mathrm{d}\, \tau < K_h < \infty$$

then the system is BIBO stable. Thus, we have proved that a sufficient condition which guarantees BIBO stability of an LTI system is that its impulse response is absolutely integrable.

As a matter of fact, this condition is also necessary, which can be demonstrated as follows. Suppose that the impulse response were not absolutely integrable, but the system were BIBO stable. Let us consider the input such that for a fixed $t x(t-\tau) = -1$ if $h(\tau) < 0$ and $x(t-\tau) = 1$ if $h(\tau) > 0$. Then the output is

$$y(t) = \int_{-\infty}^{\infty} |h(\tau)| \mathrm{d}\tau$$

which is not bounded by the assumption. This is a contradiction showing that the condition is also necessary.

Note that for an LTI causal system h(t) = 0 for t < 0 and the criterion for BIBO stability reduces to

$$\int_{0}^{\infty} \left| h(\tau) \right| \mathrm{d}\, \tau < K_h < \infty \, .$$

Graphical interpretation of continuous convolution

The convolution operation can be performed in four steps: folding, translating, multiplying, and integrating.

We explain the operations via an example. Let us consider the signals $f_1(t)$ and $f_2(t)$ as shown in Fig.1.23



We wish to determine

$$f(t) = f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau$$
 (1.44)

by its graphical interpretation.

Since $f_1(t) = 0$ for t < 0, the lower limit of integration can be replaced by 0

$$f(t) = \int_{0}^{\infty} f_1(\tau) f_2(t-\tau) \mathrm{d}\tau.$$

At first we consider the signals $f_1(\tau)$ and $f_2(\tau)$ (see Fig.1.24).





Now we take into account the signal $f_2(-\tau)$. It is obtained by folding (reflecting) $f_2(\tau)$ about the line $\tau = 0$ as depicted in Fig.1.25



Fig. 1.25. Signal $g(\tau) = f_2(-\tau)$ obtained by folding signal $f_2(\tau)$

The next step is creating $f_2(t-\tau)$ for some fixed value of t, say t = 4. Let us denote $g(\tau) = f_2(-\tau)$, then $f_2(t-\tau) = f_2(-(\tau-t)) = g(\tau-t)$. Hence, the signal $f_2(t-\tau)$ is obtained by translation (shifting) the signal $g(\tau)$ shown in Fig.1.25 by t = 4 (see Fig.1.26).



Fig. 1.26. Signal $f_2(t-\tau)$ obtained by translating signal $g(\tau)$ by t = 4

Now we perform multiplication $f_1(\tau)f_2(t-\tau)$ for t = 4. The result is shown in Fig.1.27



Fig. 1.27. Product of signals $f_1(\tau)$ and $f_2(t-\tau)$

The last step is integration

$$f(4) = \int_{0}^{\infty} f_{1}(\tau) f_{2}(4-\tau) d\tau = \int_{0}^{4} f_{1}(\tau) f_{2}(4-\tau) d\tau = 7.5.$$

Thus, the convolution at t=4 is equal to 7.5 and this is a point on the curve f(t). Selecting another t and repeating the above procedure we obtain the corresponding point of the characteristic f(t). This is shown in Figs.1.28-1.33 for t = 0, t = 1, t = 2, t = 3, t = 5, t = 6.



Fig. 1.30. Finding the convolution $f_1(t) * f_2(t)$ at t = 2



Fig. 1.33. Finding the convolution $f_1(t) * f_2(t)$ at t = 6

t	negative	0	1	2	3	4	5	greater than 5
f(t)	0	0	0.5	2	4.5	7.5	10.5	10.5

Note that for t > 5 the convolution f(t) is constant and equal to 10.5. The results of the computation are summarized in Table 1.1.

Table 1.1. The results of computation for Example 1.13

The plot of f(t) is shown in Fig.1.34.



Fig. 1.34. Plot of the convolution $f(t) = f_1(t) * f_2(t)$

The graphical method, described above, is effective if multiplication and integration of the signals can be easily performed. Generally, these operations need to be accomplished using a numerical approach. To find the convolution numerically we take into account an approximation of the convolution integral, similarly as in Section 1.7. We replace the integration by summation and the convolution is approximately given by

$$f(t) \cong \sum_{k=-\infty}^{\infty} f_1(t_k) \varepsilon f_2(t-t_k)$$
(1.45)

where we assume that ε is sufficiently small.

To illustrate the above procedure we consider the following example.

Example 1.14

Let us consider signals $f_1(t)$ and $f_2(t)$ shown in Figs. 1.35 and 1.36.



We wish to compute the convolution of $f_1(t)$ and $f_2(t)$ at t = 0.5 using the approximate formula (1.45). We choose $\varepsilon = 0.05$ and tabulate the values $t_k = k \cdot \varepsilon$, $f_1(k\varepsilon)$, and $f_2(0.5 - k\varepsilon)$ where $k = 0, 1, \dots, 10$. Next we compute the product $f_1(k\varepsilon)f_2(t-k\varepsilon)$ at any k. Finally, we add together the results and multiply by $\varepsilon = 0.05$. Note that the procedure is terminated at k = 10 ($t_k = 0.5$) because for $t_k > 0.5$ $0.5 - t_k < 0$ holds, which implies $f_2(0.5 - t_k) = 0$. The results of the computation process are summarized in Table 1.2.

k	0	1	2	3	4	5	6	7	8	9	10
t_k	0	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
$f_1(t_k)$	0	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
$f_2(0.5-t_k)$	1	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00
$f_1(t_k)f_2(0.5-t_k)$	0	0.055	0.120	0.194	0.280	0.374	0.480	0.594	0.720	0.854	1.00

Table 1.2. The results of computation for Example 1.14

We add together all numbers of the last row finding 4.670 and multiply by $\varepsilon = 0.05$. As a result we obtain $f(0.5) \approx 0.233$ whereas the correct value is f(0.5) = 0.208. To obtain more accurate values smaller ε should be chosen.

1.8.2. Discrete convolution

Definition

Let us consider the discrete-time functions $f_1(n)$ and $f_2(n)$. The discrete convolution of $f_1(n)$ and $f_2(n)$ written as

$$f(n) = f_1(n) * f_2(n)$$

is given by

$$f(n) = \sum_{k=-\infty}^{\infty} f_1(k) f_2(n-k).$$
(1.46)

Discrete convolution is commutative, i.e.

$$f_1(n) * f_2(n) = f_2(n) * f_1(n).$$

To prove this property we apply formula (1.46)

$$f_1(n) * f_2(n) = \sum_{k=-\infty}^{\infty} f_1(k) f_2(n-k)$$

and perform a change of variables by letting n - k = m, then k = n - m, $m \to \infty$ as $k \to -\infty$, and $m \to -\infty$ as $k \to \infty$. Hence, we obtain

$$f_1(n) * f_2(n) = \sum_{m=\infty}^{\infty} f_1(n-m) f_2(m) = \sum_{m=-\infty}^{\infty} f_2(m) f_1(n-m) = f_2(n) * f_1(n).$$

Applying this property to the results of Section 1.7 we state that the response of an LTI system specified by the unit sample response h(n), due to an input x(n) can be either determined using equation (1.38) repeated below

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

or using the equation

$$y(n) = \sum_{k=-\infty}^{\infty} h(k) x(n-k).$$
(1.47)

In Section 1.6.2 we defined BIBO stability of discrete-time systems. We now relate this property to the unit pulse response h(n) on LTI systems with a bounded input x(n), i.e. such that $|x(n)| < K_x$ for all n. Using equation (1.47) we obtain

$$|y(n)| = \left|\sum_{k=-\infty}^{\infty} h(k)x(n-k)\right| \le \sum_{k=-\infty}^{\infty} |h(k)||x(n-k)| \le K_x \sum_{k=-\infty}^{\infty} |h(k)|$$

Hence, if

$$\sum_{k=-\infty}^{\infty} \left| h(k) \right| < K_h < \infty$$

then the system is BIBO stable. Thus, we have proved that a sufficient condition for BIBO stability of an LTI discrete system is that its unit pulse response is absolutely summable.

It can be demonstrated that this condition is also necessary. Suppose that the unit pulse response were not absolutely summable, but the system were BIBO stable. Let us consider the input such that for a fixed n x(n-k) = -1 if h(k) < 0 and x(n-k) = 1 if h(k) > 0. Then, the output

$$y(n) = \sum_{k=-\infty}^{\infty} |h(k)|$$

is not bounded by the assumption. This is a contradiction showing that the condition is also necessary.

Note that for a causal LTI system the criterion for BIBO stability reduces to

$$\sum_{k=0}^{\infty} \left| h(k) \right| < K_h < \infty$$

Graphical interpretation of discrete convolution

Similarly as in the case of continuous convolution the discrete convolution can be performed in four steps: folding, translating, multiplying, and adding.

We will explain these operations via an example.

Example 1.15

Let us consider the convolution of the signals shown in Fig.1.37.



Fig. 1.37. Discrete signals $f_1(n)$ and $f_2(n)$

The convolution is given by (1.46) repeated below

$$f(n) = \sum_{k=-\infty}^{\infty} f_1(k) f_2(n-k).$$
(1.48)

We create the signal $f_2(-k)$ by folding the signal $f_2(k)$ about the line k = 0 (see Fig. 1.38).



Fig. 1.38. Signal $f_2(-k)$ obtained by folding signal $f_2(k)$

The next step is translating of $f_2(-k)$ for some fixed value of *n*, say n = 2. It is shown in Fig.1.39



Fig. 1.39. Signal $f_2(n-k)$ obtained by translating signal $f_2(-k)$ by n=2Now we perform multiplication $f_1(k)f_2(n-k)$, where n = 2 (see Fig.1.40)



Fig. 1.40. Product of signals $f_1(k)$ and $f_2(n-k)$ for n=2

Finally, the summation is made

$$f(n) = \sum_{k=-\infty}^{\infty} f_1(k) f_2(n-k)$$

where n = 2.

In this way we obtain f(n) = 6 for n = 2. Selecting another n and repeating the above procedure we determine corresponding f(n). Fig.1.41 shows $f_2(n-k)$ for n = 0, 1, 3, 4 and Fig.1.42 shows the corresponding products $f_1(k)f_2(n-k)$.





Fig. 1.42. Products of $f_1(k)$ and $f_2(n-k)$ for n = 0, 1, 3, 4

On the basis of Fig.1.42 we obtain f(0)=3, f(1)=5, f(3)=3, f(4)=1. It is obvious that f(n)=0 for n < 0 and n > 4. The convolution f(n) is shown in Fig.1.43.



Fig. 1.43. Plot of the convolution $f(n) = f_1(n) * f_2(n)$

Note

The discrete convolution can be also found by direct evaluating the sum given by (1.46). To illustrate this approach we consider again the discrete signals shown in Fig.1.37. Using equation (1.46) we obtain:

$$f(0) = f_1(0)f_2(0) + f_1(1)f_2(-1) + f_1(2)f_2(-2) = 3$$

$$f(1) = f_1(0)f_2(1) + f_1(1)f_2(0) + f_1(2)f_2(-1) = 5$$

$$f(2) = f_1(0)f_2(2) + f_1(1)f_2(1) + f_1(2)f_2(0) = 6$$

$$f(3) = f_1(0)f_2(3) + f_1(1)f_2(2) + f_1(2)f_2(1) = 3$$

$$f(4) = f_1(0)f_2(4) + f_1(1)f_2(3) + f_1(2)f_2(2) = 1$$

Furthermore, it is obvious that f(n) = 0 for n < 0 and for n > 4.