Electric Circuits

Technical University of Łódź
International Faculty of Engineering

Łódź 2009
# Contents

Preface.................................................................................................................. 5

1. Fundamental laws of electrical circuits......................................................... 7
   1.1. Introduction .............................................................................................. 7
   1.2. Kirchhoff’s Voltage Law (KVL)............................................................... 8
   1.3. Kirchhoff’s Current Law (KCL)............................................................... 10
   1.4. Independence of KCL equations ............................................................ 12
   1.5. Independence of KVL equations ............................................................. 13
   1.6. Tellegen’s theorem.................................................................................. 14

2. Circuit elements.................................................................................................... 16
   2.1. Resistors ................................................................................................. 16
   2.2. Independent sources .............................................................................. 18

3. Power and energy................................................................................................. 24

4. Simple linear resistive circuits.......................................................................... 26

5. Resistive circuits. DC analysis.......................................................................... 30
   5.1. Superposition theorem and its application .............................................. 30

6. Three terminal resistive circuits......................................................................... 35

7. The Thevenin-Norton theorem........................................................................... 40

8. Node method....................................................................................................... 47

9. Simple nonlinear circuits.................................................................................... 53

10. Controlled sources............................................................................................. 59

11. Capacitor ........................................................................................................... 61
   11.1. Introduction ............................................................................................ 61
   11.2. Continuity property................................................................................ 64
   11.3. Energy stored in a capacitor .................................................................. 65
   11.4. Series connection of capacitors ............................................................... 66
   11.5. Parallel connection of capacitors ............................................................ 68

12. Inductor.............................................................................................................. 69
   12.1. Introduction ............................................................................................ 69
   12.2. Continuity property................................................................................ 72
   12.3. Hysteresis............................................................................................... 73
   12.4. Energy stored in an inductor .................................................................. 74
   12.5. Series connection of inductors ................................................................. 75
   12.6. Parallel connection of inductors ............................................................... 76

13. Operational-amplifier circuits......................................................................... 78
   13.1. Description of the operational amplifier .................................................. 78
   13.2. Examples................................................................................................. 82
   13.3. Finite gain model of the operational amplifier ......................................... 85

14. First order circuits driven by DC sources....................................................... 88
   14.1. Resistor-inductor circuits ...................................................................... 88
   14.2. Resistor-capacitor circuits .................................................................... 96

15. Sinusoidal steady-state analysis ..................................................................... 104
   15.1. Preliminary discussion .......................................................................... 104
   15.2. Phasor concept ...................................................................................... 108
   15.3. Phasor formulation of circuit equations .................................................. 110
   15.4. Impedance and admittance ................................................................... 115
   15.5. Phasor diagrams.................................................................................... 122
   15.6. Effective value........................................................................................ 125

16. Power in sinusoidal steady state....................................................................... 128
   16.1. Instantaneous and average power ........................................................... 128
16.2. Complex power................................. 130
16.3. Measurement of the average power.......... 134
16.4. Theorem on the maximum power transfer..... 135

17. Resonant circuits....................................... 137
  17.1. Series resonant circuit.......................... 137
  17.2. Parallel resonant circuit....................... 144

18. Coupled inductors................................... 148
  18.1. Basic properties.................................. 148
  18.2. Connections of coupled inductors.............. 151
  18.3. Ideal transformer................................ 156

19. Three-phase systems............................... 159
  19.1. Introduction.................................... 159
  19.2. Y-connected systems........................... 161
  19.3. Three-phase systems calculations............. 164
  19.4. Power in three-phase circuits............... 168

Reference books........................................ 174
Preface

This book presents an introductory treatment of electric circuits and is intended to be used as a textbook for students, during the junior years, at the International Faculty of Engineering of the Technical University of Łódź. The book covers most of the material taught in conventional circuit courses and gives the fundamental concepts required to understand and tackle the electrical engineering problems. Its prerequisites are the basic calculus, complex numbers, and some familiarity with integral calculus and linear differential equations, which are desirable but not essential. The objective of the book is to feature theories and concepts of fundamental importance in electrical engineering that are amenable to a broad range of applications.

The book includes a large number of examples. They are provided to illustrate the concepts and to make the theory more clear. On each page there is a blank area where a student can note down comments, explanations, and additional examples discussed during the lectures.

The book can be thought of as consisting of three parts. Part 1 (Chapters 1-10, 13) introduces many basic concepts, laws, and principles related to electric circuits. In addition, different methods of the DC analysis of resistive circuits are studied in detail. Part 2 (Chapters 11-12, 14) deals with simple linear dynamic circuits and their components. The transient analysis of the first order circuits is considered. Part 3 (Chapters 15 to 19) focuses on sinusoidal circuits in the steady-state and discusses many different aspects of AC analysis. At the end of this part, three-phase systems are introduced and analysed.

I gratefully acknowledge the support and encouragement of Dr. Tomasz Saryusz-Wolski, Head of the International Faculty of Engineering of the Technical University of Łódź.

Łódź, 2009

Michał Tadeusiewicz
1. Fundamental laws of electrical circuits

1.1 Introduction

An electric circuit is an interconnection of electric devices (elements) by conducting wires. Figure 1.1 shows a circuit consisting of a voltage source, two resistors, a transistor, a capacitor, and a transformer. Any junction in the circuit where terminals of the elements are joined together is called a node. On the circuit diagrams they are marked with dots.

![Fig. 1.1. An example of a circuit](image)

In the circuits we consider currents flowing through the elements (branches) and voltages between any two nodes. The unit for voltage is the volt (V), whereas the unit for current is the ampere (A). Figure 1.2 shows the reference direction of current $i$ and voltage $v$ represented by arrows.

![Fig. 1.2. Reference directions of current $i$ and voltage $v$](image)
If at some time current is positive, then it flows into the element by node 1. If the current is negative it flows out of the element by node 1. The reference direction of the voltage across the element is represented by an arrow $v$. If at some time voltage is positive, it means that the electric potential of node 1 is larger than the electric potential of node 2. If it is negative then the electric potential of node 1 is smaller than the electric potential of node 2. The reference direction of each current and each voltage can be assigned arbitrarily. When they are chosen as shown in Fig. 1.2, we say that we have chosen associated reference directions. This is the convention we will follow throughout the whole course.

1.2 Kirchhoff’s Voltage Law (KVL)

The fundamental laws governing electric circuits are Kirchhoff’s voltage and current laws and Tellegen’s theorem. In an electric circuit we consider a path traversing some branches in succession. If the starting node of a path is the same as the ending node, the path is called a loop.

**Kirchhoff’s Voltage Law**

For any electric circuit, for any of its loop, and at any time, the algebraic sum of the branch voltages around the loop is equal to zero.

To write KVL equation we select a loop and assume arbitrarily its reference direction, clock-wise or counter clock-wise. Next we assign the plus sign to the branch voltages whose reference directions agree with that of the loop and the minus sign to the others.

**Example 1.1**
KVL equation for the loop 1, 2, 3 in the circuit shown in Fig. 1.3:

$$v_1(t) - v_2(t) + v_3(t) = 0 .$$

KVL equation for loop 1, 4, 5, 7:

$$-v_1(t) + v_4(t) + v_5(t) - v_7(t) = 0 .$$
KVL can also be expressed in terms of voltages between nodes creating a closed node sequence. A node sequence is called a closed node sequence if it starts and ends at the same node.

**Kirchhoff's Voltage Law (general version)**

For any electric circuit, for any closed node sequence, and for any time, the algebraic sum of all node-to-node voltages around the chosen closed node sequence is equal to zero.

**Example 1.2**
Let us consider closed node sequences: 1, 2, 5, 4, 1 and 1, 2, 3, 6, 5, 4, 1.

KVL equations:
1, 2, 5, 4, 1: \(-v_2 - v_{2.5} + v_3 + v_1 = 0\).
1, 2, 3, 6, 5, 4, 1: \(-v_2 + v_4 + v_6 - v_5 + v_3 + v_1 = 0\).

1.3 Kirchhoff’s Current Law (KCL)

Another fundamental law governing electric circuits is Kirchhoff’s Current Law, as follows.

For any electric circuit, for any of its nodes, and at any time the algebraic sum of all the branch currents meeting at the node is zero.

In the algebraic sum we assign the plus sign to the currents leaving the node and the minus sign to the currents entering the node.

Example 1.3

![Fig. 1.5. An example circuit for illustrating KCL](image)

1: \(i_1(t) - i_2(t) + i_3(t) = 0\),
or simply
\(i_1 - i_2 + i_3 = 0\).

3: \(-i_3 - i_4 = 0\).
To formulate KCL in a more general form we consider gaussian surface defined as a balloon-like closed surface, as illustrated in Fig. 1.6.

![Fig. 1.6. The gaussian surface](image)

**KCL (general version)**

For all circuits, for all gaussian surfaces, for all times $t$, the algebraic sum of all currents crossing the gaussian surface at time $t$ is equal to zero. In the algebraic sum we assign the plus sign to the currents leaving the gaussian surface and the minus sign to the currents entering the surface.

**Example 1.4**

In the circuit shown in Fig. 1.6 we write KCL equation

$$-i_1 + i_2 - i_3 = 0.$$
1.4 Independence of KCL equations

For a given circuit we can write many KCL equations. Hence, the question arises how many of them are linearly independent.

![Example Graph](image)

Fig. 1.7. An example graph

To answer this question we consider the graph shown in Fig. 1.7 and write KCL equations at each node

\[
\begin{align*}
i_1 - i_2 - i_3 &= 0 , \\
-i_1 + i_2 + i_4 &= 0 , \\
i_3 - i_4 &= 0 .
\end{align*}
\]

If we add the first two equations together, we obtain

\[
-i_3 + i_4 = 0 .
\]

Multiplying both sides of this equation by \((-1)\) yields

\[
i_3 - i_4 = 0 ,
\]

which is exactly the third equation.
It means that the third equation is a linear combination of the first two equations. Thus, not each equation brings new information not contained in the others and at least one equation repeats the information contained in the others. However, if we reject the third equation, then the remaining ones are linearly independent. Thus, the third equation is redundant, it is useless and can be discarded. Generally, the following independence property of KCL equations holds. For any graph with $n$ nodes KCL equations for any $(n-1)$ of these nodes form a set of $(n-1)$ linearly independent equations.

### 1.5 Independence of KVL equations

Similarly as in the case of KCL equations the question arises how to write a set of linearly independent KVL equations. The simplest answer is as follows. We write KVL equations selecting the loops so that any equation contains at least one voltage that has not been included in any of the previous equations.

It can be shown that for a circuit having $b$ branches and $n$ nodes $b - n + 1$ linearly independent equations can be formulated.

![Fig. 1.8. A graph for illustrating independence of KVL equations](image)

**Example 1.5**

Let us consider the graph shown in Fig. 1.8. In this graph we write linearly independent KVL equations using the provided rule. As a result we obtain the following set of equations
\[ v_1 + v_7 + v_2 = 0 \, , \]

\[ -v_2 - v_3 - v_4 = 0 \, , \]

\[ v_4 + v_5 + v_6 = 0 \, , \]

\[ v_7 + v_6 + v_8 = 0 \, . \]

### 1.6 Tellegen’s theorem

Let us consider a graph having \( b \) branches and \( n \) nodes. Let us use the associated reference directions.

**Tellegen’s theorem**

Let \( \{i_1, i_2, \ldots, i_b\} \) be any set of branch currents satisfying KCL at any node and let \( \{v_1, v_2, \ldots, v_b\} \) be any set of branch voltages satisfying KVL at any loop. Then it holds

\[
\sum_{k=1}^{b} v_k i_k = 0 \, .
\]

Note that the set of branch currents and the set of branch voltages are associated with the given graph but not necessarily with the same circuit. For example, let us consider the graph shown in Fig. 1.9 and two different circuits depicted in Figs 1.10 and 1.11 having this graph.

Tellegen’s theorem enables us to write the following equations:

\[ v_1 \tilde{i}_1 + v_2 \tilde{i}_2 + v_3 \tilde{i}_3 + v_4 \tilde{i}_4 = 0 \, , \]

\[ \tilde{v}_1 \tilde{i}_1 + \tilde{v}_2 \tilde{i}_2 + \tilde{v}_3 \tilde{i}_3 + \tilde{v}_4 \tilde{i}_4 = 0 \, , \]

\[ v_1 \tilde{v}_1 + v_2 \tilde{v}_2 + v_3 \tilde{v}_3 + v_4 \tilde{v}_4 = 0 \, , \]

\[ \tilde{v}_1 \tilde{v}_1 + \tilde{v}_2 \tilde{v}_2 + \tilde{v}_3 \tilde{v}_3 + \tilde{v}_4 \tilde{v}_4 = 0 \, . \]
Fig. 1.9. A graph having three nodes and four branches

Fig. 1.10. A circuit having the graph of Fig. 1.9

Fig. 1.11. A circuit having the graph of Fig. 1.9
2. Circuit elements

The components used to build electric circuits are called circuit elements. In this section we define simple two-terminal circuit elements: a resistor, independent voltage and current sources.

2.1 Resistors

An element is said to be a resistor if its voltage-current relation is of algebraic type. This relation is represented graphically by a curve in $v-i$ plane, called the characteristic of the resistor. Any resistor can be classified as linear or nonlinear.

A resistor is called linear if its characteristic is a straight line through the origin (see Fig. 2.1).

![Fig. 2.1. Characteristic of a linear resistor](image)

It is described by the equation

$$v = Ri$$  \hspace{1cm} (2.1)

or

$$i = Gv$$  \hspace{1cm} (2.2)
where \( G = 1/R \). Equation (2.1) is known as Ohm’s law, \( R \) is called the resistance and \( G \) is called the conductance. The unit for resistance is the ohm (\( \Omega \)) and for conductance the siemens (\( S \)). The symbol of a linear resistor is shown in Fig. 2.2.

![Fig. 2.2. The symbol of a linear resistor](image)

Any resistor whose \( v - i \) characteristic is not a straight line through the origin is classified as a nonlinear resistor. A typical example of a nonlinear resistor is a diode, described by the equation

\[
i = K(e^{\lambda v} - 1), \tag{2.3}
\]

where \( K \) and \( \lambda \) are positive constants. Fig. 2.3 shows the symbol and the characteristic of a semiconductor diode.

![Fig. 2.3. Symbol and characteristic of a diode](image)
A general symbol of any nonlinear resistor is depicted in Fig. 2.4.

![Fig. 2.4. General symbol of a nonlinear resistor](image)

### 2.2 Independent sources

**Voltage source**

An element is called a voltage source if it maintains a prescribed voltage $v_S(t)$ between its terminals for any current flowing through the source. Consequently, a voltage source maintains a prescribed voltage $v_S(t)$ between its terminals in an arbitrary circuit to which it is connected.

The symbol of the voltage source is shown in Fig. 2.5.

![Fig. 2.5. The symbol of a voltage source](image)

Generally, the prescribed voltage is a time varying signal $v_S(t)$. In a special case, it is constant $V_S$, called a DC voltage source. In such a case, the characteristic expressing the voltage between the terminals of the source in terms of the current flowing through the source is a horizontal line, as shown in Fig. 2.6.

![Fig. 2.6. Characteristic of a DC voltage source](image)
The defined voltage source is an ideal element not encountered in the physical world. A real voltage source can be represented by an equivalent circuit shown in Fig. 2.7.

![Fig. 2.7. Model of a real voltage source](image)

Certain devices have $R_S$ very small and can quite effectively be approximated by the ideal voltage source. Let us consider a real voltage source terminated by a load, as shown in Fig. 2.8.

![Fig. 2.8. A real voltage source terminated by a load](image)

Using KVL and Ohm’s law we write the equation

$$v = V_S - R_S i,$$

that describes, on the $i - v$ plane, the straight line, shown in Fig. 2.9. This line is a characteristic of the real voltage source, and is called a load line.
Current source

A current source is an element which maintains a prescribed current $i_s(t)$ for any voltage $v(t)$ between its terminals. Consequently, a current source maintains a prescribed current $i_s(t)$ in an arbitrary circuit to which it is connected. The symbol of a current source is shown in Fig. 2.10.

![Current Source Symbol](image)

Fig. 2.10. Symbol of a current source

If the prescribed current is constant, $i_s(t) = I_s$ the current source is called a DC current source. Its $i - v$ characteristic is shown in Fig. 2.11.
The defined current source is an ideal element. A real current source can be represented by the circuit shown in Fig. 2.12.

Let us consider a real DC current source terminated by a load, as shown in Fig. 2.13.
Applying KCL at the top node we write

\[ i + i_R - I_S = 0 \]  \hspace{1cm} (2.5)

Since

\[ i_R = \frac{v}{R_S} \]  \hspace{1cm} (2.6)

then

\[ i = I_S - \frac{v}{R_S} \]  \hspace{1cm} (2.7)

Equation (2.7) describes \( v - i \) characteristic of a real DC source, as shown in Fig. 2.14. This characteristic is a straight line, called a load line.
Let us multiply both sides of equation (2.7) by $R_S$ and rearrange this equation as follows

$$v = R_S I_S - R_S i.$$  

(2.8)

By denoting $V_S = R_S I_S$, we obtain equation (2.4) that describes a real voltage source. Thus, if $V_S = R_S I_S$, then the real current source and the real voltage source are equivalent.
3. Power and energy

Let us consider a circuit and draw two wires from this circuit. As a result we obtain a two-terminal circuit called a one-port. If the one-port is supplied with a source, the current $i(t)$ flows into the one-port by terminal A and the same current $i(t)$ flows out of the one-port by terminal B. Therefore, we indicate only one current $i(t)$, as shown in Fig. 3.1. The voltage $v(t)$ between the terminals is also indicated.

![Fig. 3.1. A one-port with indicated the port voltage and port current](image)

The current $i(t)$ and the voltage $v(t)$ are called a port current and a port voltage, respectively.

The instantaneous power entering the one-port is equal to the product of the port voltage and port current

$$p(t) = v(t)i(t),$$

(3.1)

where $v(t)$ is in volts, $i(t)$ in amperes and $p(t)$ in watts (abbreviated to W).

The energy delivered to the one-port from time $t_0$ to $t$ is given by the equation

$$w(t_0, t) = \int_{t_0}^{t} p(\tau) d\tau = \int_{t_0}^{t} v(\tau) i(\tau) d\tau,$$
where a variable \( \tau \) means time. Hence, it holds

\[
\frac{dw}{dt} = p(t) .
\]

If the one-port is a linear resistor, specified by \( R \) or \( G \), then

\[
P(t) = v(t)i(t) = Ri^2(t) , \quad (3.2)
\]

\[
P(t) = v(t)i(t) = Gv^2(t) . \quad (3.3)
\]

Thus, the instantaneous power is, in this case, nonnegative for all \( t \).

If the one-port is a nonlinear resistor, represented by a \( v-i \) characteristic located in the first and third quadrants only, then \( v(t)i(t) \geq 0 \) and the power entering the resistor is nonnegative, its energy is a nondecreasing function of time and the resistor consumes the energy. Such a resistor is called passive. If some parts of the \( v-i \) characteristic lie in the second or third quadrant, then for some \( t \), \( v(t)i(t) < 0 \), the power entering the resistor is negative and the resistor delivers energy to the outside world. Such a resistor is called active.
4. Simple linear resistive circuits

Circuits consisting of resistors and sources are classified as resistive circuits. In particular, they can be supplied with DC sources only. The analysis of such a class of circuits is called DC analysis.

Let us consider a circuit consisting of two linear resistors connected in series, as shown in Fig. 4.1.

\[
R_2 \quad R_1
\]

\[v_2 \quad v_1 \quad v\]

Fig. 4.1. Two resistors connected in series

To analyse this circuit we apply the KVL and Ohm’s law. Since the same current traverses both resistors we write

\[v = v_1 + v_2 = R_1 i + R_2 i = (R_1 + R_2) i . \]  \hspace{1cm} (4.1)

Hence, we have

\[\frac{v}{i} = R_1 + R_2 . \]  \hspace{1cm} (4.2)

Equation (4.2) states that the series connection of two linear resistors is equivalent to resistor \(R\),

\[R = R_1 + R_2 . \]  \hspace{1cm} (4.3)

Voltages across the resistors are specified by the equations:

\[v_1 = R_1 i = \frac{R_1}{R_1 + R_2} v , \quad v_2 = R_2 i = \frac{R_2}{R_1 + R_2} v . \]
Hence, it follows the relation

\[
\frac{v_1}{v_2} = \frac{R_1}{R_2},
\]

(4.4)

which states that the series connection of resistors \(R_1\) and \(R_2\) can be considered as a voltage divider. The voltage \(v\) is divided in proportion to \(R_1\) and \(R_2\). Formula (4.3) can be directly generalized to the series connection of \(n\) linear resistors, \(R_1, \ldots, R_n\)

\[
R = R_1 + \cdots + R_n.
\]

(4.5)

Figure 4.2 shows the circuit consisting of two linear resistors connected in parallel.

Volatges across resistors \(R_1\) and \(R_2\) are identical and equal to \(v\). The current \(i\) according to KCL satisfies the equation:

\[
i = i_1 + i_2.
\]

(4.6)

Using Ohm’s law

\[
i_1 = \frac{v}{R_1}, \quad i_2 = \frac{v}{R_2}
\]

(4.7)
we obtain

\[ i = \left( \frac{1}{R_1} + \frac{1}{R_2} \right) v, \quad (4.8) \]

or

\[ v = R i \quad (4.9) \]

where

\[ R = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} = \frac{R_1 R_2}{R_1 + R_2}. \quad (4.10) \]

Thus, two resistors connected in parallel are equivalent to the resistor \( R \) specified by (4.10).

Using (4.7), (4.9), and (4.10) we write

\[ i_1 = \frac{v}{R_1} = \frac{R_2}{R_1 + R_2} i, \quad i_2 = \frac{v}{R_2} = \frac{R_1}{R_1 + R_2} i. \]

Hence, it holds

\[ \frac{i_1}{i_2} = \frac{R_2}{R_1}. \quad (4.11) \]

Thus, the parallel connection of resistors \( R_1 \) and \( R_2 \) can be considered as a current divider, where the currents are divided according to equation (4.11).

Equation (4.10) can be directly generalised to the circuit consisting of \( n \) resistors \( R_1, \ldots, R_n \) connected in parallel

\[ R = \frac{1}{\frac{1}{R_1} + \cdots + \frac{1}{R_n}}. \quad (4.12) \]

Figure 4.3 shows a three-terminal resistor called a potentiometer. Terminal 3 called a wiper, can be shifted along the resistor \( R_p \), dividing it into \( R_x \) and \( R_y \).
Let us consider the circuit shown in Fig. 4.4, containing a potentiometer. The circuit can be considered as a series connection of resistor $R_x$ and parallel connected resistors $R_y$ and $R$. Hence, the resistance faced by the voltage source is

$$R = R_x + \frac{R_y R}{R_y + R}. \quad (4.12)$$

Formula (4.12) enables us to find the current $i$

$$i = \frac{v}{R_x + \frac{R_y R}{R_y + R}}. \quad (4.13)$$

and then voltage $v_y$

$$v_y = i \cdot \frac{R_y R}{R_y + R} = \frac{R_y R}{R_x (R_y + R) + R_y R} v. \quad (4.14)$$
5. Resistive circuits. DC analysis

In this section we study circuits consisting of linear resistors and independent DC sources. We formulate some theorems governing these circuits, which enables us to analyse them efficiently.

5.1 Superposition theorem and its application

Let us consider a linear circuit driven by \( n \) voltage sources
\[
V_{S_1}, V_{S_2}, ..., V_{S_n},
\]
and \( m \) current sources
\[
I_{S_1}, I_{S_2}, ..., I_{S_m}.
\]
The superposition theorem states that any branch current and any branch voltage in this circuit is given by the expression of the form
\[
h_1V_{S_1} + h_2V_{S_2} + ... + h_nV_{S_n} + k_1I_{S_1} + k_2I_{S_2} + ... + k_mI_{S_m},
\]
where coefficients \( h_j \) \((j = 1, ..., n)\) and \( k_j \) \((j = 1, ..., m)\) are constants and depend only on circuit parameters.

In other words, any branch current and any branch voltage is a linear combination of the voltage and current sources.

Example 5.1
Let us consider a circuit consisting of linear resistors, a single voltage source and a single current source. We extract from this circuit the sources and an arbitrary resistor, as shown in Fig. 5.1. We wish to find the current \( i \) flowing through resistor \( R \) using the superposition theorem.
According to the superposition theorem
\[ i = hV_s + kI_s . \] (5.2)

Let \( \tilde{i} = hV_s \) and \( \tilde{i} = kI_s \), then
\[ i = \tilde{i} + \tilde{i} . \] (5.3)

Note that \( i = \tilde{i} \) if \( I_s = 0 \) and \( i = \tilde{i} \) if \( V_s = 0 \).

If \( I_s = 0 \), then the branch containing the current source can be replaced by an open circuit (see Fig. 5.2). Thus, \( \tilde{i} \) is the current flowing through resistor \( R \) in the circuit with the current source set to zero (removed). If \( V_s = 0 \), then the branch containing the voltage source can be replaced by a short circuit (see Fig. 5.3). Thus, \( \tilde{i} \) is the current flowing through resistor \( R \) in the circuit with the voltage source set to zero (short-circuited). The current \( \tilde{i} \) can be regarded as a response of the circuit due to the voltage source \( V_s \) acting alone. The current \( \tilde{i} \) can be considered a response of the circuit due to the current source \( I_s \) acting alone.
Generally, the response of the circuit due to several voltage and current sources is equal to the sum of the responses due to each source acting alone, that is with all other voltage sources replaced by short circuits and all other current sources replaced by open circuits.

**Example 5.2**
Let us consider the circuit shown in Fig. 5.4, driven by a voltage source $V_S$ and a current source $I_S$. We apply the superposition theorem to find voltage $v_2$. 

**Notes**
First we set the current source to zero. As a result, the circuit is driven only by the voltage source $V_S$ (see Fig. 5.5).

![Fig. 5.5. Circuit of Fig. 5.4 driven only by the voltage source $V_S$](image)

In this circuit the same current $\tilde{i}_2$ traverses all the resistors. Hence, we find:

$$\tilde{i}_2 = \frac{V_S}{R_1 + R_2 + R_3},$$

whereas the voltage $\tilde{v}_2$ is, according to Ohm’s law, given by the equation:

$$\tilde{v}_2 = R_2 \tilde{i}_2 = \frac{R_2}{R_1 + R_2 + R_3} V_S.$$

Now we set the voltage source $V_S$ to zero, obtaining the circuit shown in Fig. 5.6.

![Fig. 5.6. Circuit of Fig. 5.4 driven only by the current source $I_S$](image)
In this circuit the resistance faced by the current source equals
\[
\frac{(R_1 + R_3)R_2}{R_1 + R_3 + R_2}.
\]

The product of this resistance and the current \( I_S \) is voltage \( \tilde{v}_2 \)
\[
\tilde{v}_2 = \frac{(R_1 + R_3)R_2}{R_1 + R_3 + R_2} I_S.
\]

The superposition theorem leads to the equation
\[
v_2 = \bar{v}_2 + \tilde{v}_2 = \frac{R_2}{R_1 + R_3 + R_2} V_S + \frac{(R_1 + R_3)R_2}{R_1 + R_3 + R_2} I_S.
\]
6. Three terminal resistive circuits

The three-terminal circuits shown in Figs 6.1 and 6.2 consist of three linear resistors. The circuit of Fig. 6.1 is called a \( \text{Y} \) circuit, whereas the one depicted in Fig. 6.2 is called a \( \text{Δ} \) circuit.

Fig. 6.1. \( \text{Y} \) circuit

Fig. 6.2. \( \text{Δ} \) circuit
In a Y circuit it holds

\[ i_3 = i_1 + i_2 . \]  \hspace{1cm} (6.1)

Hence, we find

\[ v_1 = v_{R_1} + v_{R_2} = R_1 i_1 + R_3 (i_1 + i_2) = (R_1 + R_3) i_1 + R_3 i_2 , \]  \hspace{1cm} (6.2)

\[ v_2 = v_{R_2} + v_{R_3} = R_2 i_2 + R_3 (i_1 + i_2) = R_3 i_1 + (R_2 + R_3) i_2 . \]  \hspace{1cm} (6.3)

To express \( v_1 \) and \( v_2 \) in terms of \( i_1 \) and \( i_2 \) in the \( \Delta \) circuit, we apply current sources \( i_1 \) and \( i_2 \) to this circuit, as shown in Fig. 6.3.

![Fig. 6.3. \( \Delta \) circuit driven by two current sources](image)

To find \( v_1 \) and \( v_2 \) we use the superposition theorem. First, we set current source \( i_2 \) to zero (see Fig. 6.4) and compute voltages \( \bar{v}_1 \) and \( \bar{v}_2 \)
\[ \tilde{v}_1 = \frac{R_{31}(R_{12} + R_{23})}{R_{31} + R_{12} + R_{23}} i_1 , \]  
\[ (6.4) \]

\[ \tilde{v}_2 = \frac{\tilde{v}_1 R_{23}}{R_{12} + R_{23}} = \frac{R_{31} R_{23}}{R_{31} + R_{12} + R_{23}} i_1 . \]  
\[ (6.5) \]

Fig. 6.4. The circuit shown in Fig. 6.3 with \( i_2 = 0 \)

Next, we set current source \( i_1 \) to zero (see Fig. 6.5) and compute \( \tilde{v}_1 \) and \( \tilde{v}_2 \)

\[ \tilde{v}_2 = \frac{R_{23}(R_{12} + R_{31})}{R_{31} + R_{12} + R_{23}} i_2 , \]  
\[ (6.6) \]

\[ \tilde{v}_1 = \frac{\tilde{v}_2 R_{31}}{R_{12} + R_{31}} = \frac{R_{23} R_{31}}{R_{23} + R_{12} + R_{31}} i_2 . \]  
\[ (6.7) \]
Fig. 6.5. The circuit shown in Fig 6.3 with $i_1 = 0$

According to the superposition theorem we obtain

$$v_1 = \tilde{v}_1 + \tilde{v}_1 = \frac{R_{31}(R_{12} + R_{23})}{R_{31} + R_{12} + R_{23}} i_1 + \frac{R_{23}R_{31}}{R_{23} + R_{12} + R_{31}} i_2,$$

(6.8)

$$v_2 = \tilde{v}_2 + \tilde{v}_2 = \frac{R_{31}R_{23}}{R_{31} + R_{12} + R_{23}} i_1 + \frac{R_{23}(R_{12} + R_{31})}{R_{23} + R_{12} + R_{31}} i_2.$$

(6.9)

The $Y$ and $\Delta$ circuits are said to be equivalent if the sets of equations (6.4)-(6.5) and (6.8)-(6.9) are identical. In such a case, the corresponding coefficients of these equations are equal, i.e.

$$R_1 + R_3 = \frac{R_{31}(R_{12} + R_{23})}{R_{31} + R_{12} + R_{23}}.$$

(6.10)

$$R_3 = \frac{R_{23}R_{31}}{R_{23} + R_{12} + R_{31}}.$$

(6.11)

$$R_2 + R_3 = \frac{R_{23}(R_{12} + R_{31})}{R_{23} + R_{12} + R_{31}}.$$

(6.12)
Solving this set of equations for $R_1$, $R_2$, $R_3$ we find

$$R_1 = \frac{R_{12}R_{31}}{R_{12} + R_{23} + R_{31}}, \quad (6.13)$$

$$R_2 = \frac{R_{12}R_{23}}{R_{12} + R_{23} + R_{31}}, \quad (6.14)$$

$$R_3 = \frac{R_{23}R_{31}}{R_{12} + R_{23} + R_{31}}. \quad (6.15)$$

Formulas (6.13)-(6.15) give the resistances of the $Y$ circuit which is equivalent to the $\Delta$ circuit. Solving the set of equations (6.12)-(6.14) for $R_{12}$, $R_{23}$, $R_{31}$ we find

$$R_{12} = R_1 + R_2 + \frac{R_1R_2}{R_3}, \quad (6.16)$$

$$R_{23} = R_2 + R_3 + \frac{R_2R_3}{R_4}, \quad (6.17)$$

$$R_{31} = R_3 + R_1 + \frac{R_3R_1}{R_2}. \quad (6.18)$$

Formulas (6.16)-(6.18) give the resistances of the $\Delta$ circuit which is equivalent to the $Y$ circuit.

The $Y$ circuit is said to be balanced if $R_1 = R_2 = R_3 = R_Y$. Using in such a case, (6.16)-(6.18), we obtain $R_{12} = R_{23} = R_{31} = R_\Delta$, where

$$R_\Delta = 3R_Y. \quad (6.19)$$
The Thevenin-Norton theorem is a very important law governing linear resistive circuits. It can be regarded as two equivalent theorems.

Let us consider an arbitrary one-port consisting of linear resistors and independent sources, as shown in Fig. 7.1.

![Fig. 7.1. A linear resistive one-port](image)

**The Thevenin theorem**

Any linear resistive one-port can be replaced by a series connection of a resistor \( R_{eq} \) and a voltage source \( V_{OC} \), where \( R_{eq} \) is an input resistance across the one-port after all sources inside it are set to zero, \( V_{OC} \) is a voltage across the terminals of the one-port when the port is left open-circuited.

The equivalent Thevenin circuit is shown in Fig. 7.2. The elements of this circuit \( R_{eq} \) and \( V_{OC} \) can be determined as illustrated in Figs 7.3 and 7.4.
The Norton theorem
Any linear resistive one-port can be replaced by a parallel connection of a linear resistor \( R_{eq} \) and a current source \( I_{SC} \).

\( R_{eq} \) is defined as in the Thevenin theorem. \( I_{SC} \) is a current flowing through the short-circuited one-port.
The equivalent Norton circuit is shown in Fig. 7.5, whereas Fig. 7.6 shows the circuit enabling us to find $I_{sc}$.

Fig. 7.5. The equivalent Norton circuit

The circuit shown in Fig. 7.6 can be replaced, on the basis of Thevenin’s theorem, by an equivalent circuit shown in Fig. 7.7.

Fig. 7.7. The circuit equivalent to the circuit of Fig. 7.6
Using KVL and Ohm’s law we write

\[ V_{0C} - R_{eq}I_{SC} = 0. \]  

(7.1)

Hence, it follows the equation

\[ R_{eq} = \frac{V_{0C}}{I_{SC}}, \]  

(7.2)

showing the relation between \( R_{eq} \), \( V_{0C} \), and \( I_{SC} \).

**Proof of the Thevenin theorem**

Let us consider a resistive one-port, shown in Fig. 7.8, containing \( n \) voltage sources \( V_{S1}, V_{S2}, \ldots, V_{Sn} \) and \( m \) current sources \( I_{S1}, I_{S2}, \ldots, I_{Sm} \).

![Fig. 7.8. A resistive one-port](image1)

![Fig. 7.9. The one-port of Fig. 7.8 supplied with a current source](image2)

We connect an additional current source to the one-port, as shown in Fig. 7.9.
On the basis of the superposition theorem we obtain

\[ v = \sum_{j=1}^{n} h_j V_{sj} + \sum_{j=1}^{m} k_j I_{sj} + k_0 i. \]  

(7.3)

If \( i = 0 \), then the port terminals are open-circuited and \( v = V_{0C} \). Hence, we have

\[ V_{0C} = \sum_{j=1}^{n} h_j V_{sj} + \sum_{j=1}^{m} k_j I_{sj}. \]  

(7.4)

If we set all the sources inside the one-port to zero, that is \( V_{S_1} = V_{S_2} = \ldots = V_{S_n} = 0 \), \( I_{S_1} = I_{S_2} = \ldots = I_{S_n} = 0 \), then equation (7.3) reduces to

\[ v = k_0 i, \]  

(7.5)

hence,

\[ k_0 = \frac{v}{i} = R_{eq}. \]  

(7.6)

and equation (7.3) can be rewritten in the form

\[ v = V_{0C} + R_{eq} i, \]  

(7.7)

where \( V_{0C} \) and \( R_{eq} \) are defined as in the Thevenin theorem. Equation (7.7) describes the circuit depicted in Fig. 7.8, being Thevenin’s equivalent circuit.

Fig. 7.8. The Thevenin equivalent circuit
Note that the Thevenin circuit shown in Fig. 7.8 is equivalent to the circuit depicted in Fig. 7.9, being the Norton circuit. Thus, the Thevenin and Norton circuits are equivalent.

\[
\frac{V_{OC}}{R_{eq}} = \frac{V_{OC}}{R_{eq}} = I_{SC}
\]

Fig. 7.9. The circuit equivalent to the circuit shown in Fig. 7.8

**Example**
Let us consider the one-port shown in Fig. 7.10.

The equivalent Thevenin circuit is shown in Fig. 7.11. We find the elements \( V_{OC} \) and \( R_{eq} \) of this circuit.

According to Thevenin’s theorem \( R_{eq} \) is the input resistance of the one-port shown in Fig. 7.12.
Hence, we have

\[ R_{eq} = \frac{R_1R_2}{R_1 + R_2} \]  \hspace{1cm} (7.8)

To find \( V_{0C} \) we consider a circuit with open-circuited terminals shown in Fig. 7.13. In this circuit the same current \( i \) traverses all the elements. To find this current we apply KVL and Ohm’s law

\[ i = \frac{E_1 - E_2}{R_1 + R_2} \]  \hspace{1cm} (7.9)

Since the current \( i \) flows through resistor \( R_2 \), we have

\[ V_{0C} = E_2 + R_2i = E_2 + R_2 \frac{E_1 - E_2}{R_1 + R_2} = \frac{E_2R_1 + E_1R_2}{R_1 + R_2} \]  \hspace{1cm} (7.10)
8. Node method

Kirchhoff’s laws and Ohm’s law enable us to analyse simple resistive circuits. However, such an approach is inefficient in the case of more complex circuits. In this Section we develop a general, very useful and commonly applied method, called the node method.

To explain this method we consider a circuit having \( n \) nodes and introduce a concept of node-to-datum voltage. For this purpose we choose arbitrarily one of these nodes as a datum node. The potential of this node is set to zero, hence, it is grounded. For the remaining \( n-1 \) nodes we introduce node-to-datum voltages (or simply node voltages) \( e_1, \ldots, e_{n-1} \), between these nodes and the datum. The reference directions of the node voltages are shown in Fig. 8.1.

![Fig. 8.1. Reference directions of the node voltages](image)

It is easy to see that voltage between any two nodes \( k \) and \( j \) can be expressed in terms of node voltages \( e_k \) and \( e_j \).

\[
v_{kj} = e_k - e_j .
\] (8.1)
The idea of the node method will be explained using the circuit shown in Fig. 8.3, where the current sources $i_{S_1}$, $i_{S_2}$ and the conductances $G_1 - G_5$ are given.
We choose the bottom node as a reference, introduce node voltages and write KCL equations at nodes 1, 2, 3

\[
\begin{align*}
1 & \quad i_1 + i_4 - i_{S_1} + i_{S_2} = 0, \\
2 & \quad -i_2 - i_4 - i_5 = 0, \\
3 & \quad i_3 + i_5 - i_{S_2} = 0.
\end{align*}
\] (8.2)

Next, we express the branch currents in terms of node voltages

\[
\begin{align*}
i_1 &= G_1 v_1 = G_1 e_1, \\
i_2 &= G_2 v_2 = G_2 (-e_2) = -G_2 e_2, \\
i_3 &= G_3 v_3 = G_3 e_3, \\
i_4 &= G_4 v_4 = G_4 (e_1 - e_2), \\
i_5 &= G_5 v_5 = G_5 (e_3 - e_2).
\end{align*}
\] (8.3)

and substitute them into (8.2)

\[
\begin{align*}
G_1 e_1 + G_4 (e_1 - e_2) &= i_{S_1} - i_{S_2}, \\
-G_4 (e_1 - e_2) + G_2 e_2 - G_5 (e_3 - e_2) &= 0, \\
G_3 e_3 + G_5 (e_3 - e_2) &= i_{S_2}.
\end{align*}
\] (8.4)

Finally, we rearrange equations (8.4) as follows

\[
\begin{align*}
(G_1 + G_4) e_1 - G_4 e_2 &= i_{S_1} - i_{S_2}, \\
-G_4 e_1 + (G_4 + G_2 + G_5) e_2 - G_5 e_3 &= 0, \\
-G_5 e_2 + (G_3 + G_5) e_3 &= i_{S_2}.
\end{align*}
\] (8.5)

The set of node equations (8.5) contains three unknowns $e_1$, $e_2$, $e_3$. 

Notes

1
2
3
49
Let us replace the resistor $G_2$ by a voltage source $V_{S_2}$, as shown in Fig. 8.4.

![Circuit diagram](image)

**Fig. 8.4. Circuit driven by current and voltage sources**

Equations written at node 1 and 3 are the same as in the previous case. Hence, we only need to write an equation at node 2. Since the current $i_2$ cannot be expressed in terms of the branch voltage, it is considered an additional variable. Thus, we obtain the following set of equations

1. $G_1 e_1 + G_4 (e_1 - e_2) = i_{S_1} - i_{S_2}$,
2. $-G_4 (e_1 - e_2) + i_2 - G_5 (e_3 - e_2) = 0$,
3. $G_3 e_3 + G_5 (e_3 - e_2) = i_{S_2}$.

(8.6)
This is a set of three equations with four unknown variables $e_1$, $e_2$, $e_3$, $i_2$. Therefore, we add another equation of the form

$$e_2 = V_{S_2} ,$$

(8.7)

where $V_{S_2}$ is the given voltage source. Substituting (8.7) into (8.6) we eliminate the variable $e_2$ and obtain the set of three equations in three variables $e_1$, $e_3$, $i_2$

1. $G_1 e_1 + G_4(e_1 - V_{S_2}) = i_{S_1} - i_{S_2}$,
2. $-G_4(e_1 - V_{S_2}) + i_2 - G_5(e_3 - V_{S_2}) = 0$,
3. $G_3 e_3 + G_5(e_3 - V_{S_2}) = i_{S_2}$

(8.8)

Fig. 8.5. Nonlinear resistor circuit
The node method can be also applied to circuits containing nonlinear resistors. It will be explained via an example circuit shown in Fig. 8.5, including a nonlinear resistor (semiconductor diode) described by the equation

\[ i_5 = K(e^{i\alpha_5} - 1). \]  

(8.9)

To write the node equations we introduce temporarily the current \( i_5 \) as an additional variable

\[
\begin{align*}
1 & : G_1 e_1 + G_4 (e_1 - e_2) = i_{S_1} - i_{S_2}, \\
2 & : -G_4 (e_1 - e_2) + G_2 e_2 - i_5 = 0, \\
3 & : G_3 e_3 + i_5 = i_{S_2}.
\end{align*}
\]

(8.10)

Next we express \( i_5 \) in terms of the corresponding node voltages

\[ i_5 = K(e^{i\alpha_5} - 1) = K(e^{i(e_3 - e_2)} - 1) \]  

(8.11)

and substitute it into (8.10)

\[
\begin{align*}
1 & : G_1 e_1 + G_4 (e_1 - v_{S_2}) = i_{S_1} - i_{S_2}, \\
2 & : -G_4 (e_1 - e_2) + G_2 e_2 - K(e^{i(e_3 - e_2)} - 1) = 0, \\
3 & : G_3 e_3 + K(e^{i(e_3 - e_2)} - 1) = i_{S_2}.
\end{align*}
\]

(8.12)

In this way, we obtain a set of three node equations, in three unknown variables, describing the nonlinear circuit shown in Fig. 8.5.
9. Simple nonlinear circuits

In this section we analyse very simple circuits consisting of nonlinear resistors by means of a graphical approach.

**Series connection of resistors**

Figure 9.1 shows a circuit consisting of two nonlinear resistors connected in series.

![Series connection of resistors](image)

The resistors are specified by their characteristics depicted in Figs 9.2 and 9.3.

![Characteristics](image)

**Notes**

The resistors are specified by their characteristics depicted in Figs 9.2 and 9.3.
On the basis of KVL we write

\[ v = v_1 + v_2 . \]  

(9.1)

Since the resistors are connected in series, the same current traverses each of them, hence, it holds

\[ i_1 = i_2 = i . \]  

(9.2)

On the basis of these equations we can find the characteristic \( i - v \) using a graphical approach. To trace the characteristic we add the voltages \( v_1 \) and \( v_2 \) specified by the characteristics \( i - v_1 \) and \( i - v_2 \), respectively, for each value of the current \( i \).

The graphical construction is illustrated in Fig. 9.4.

![Graphical construction for finding characteristic \( i-v \)](image)

**Fig. 9.4. Graphical construction for finding characteristic \( i-v \)**

**Parallel connection of resistors**

Figure 9.5. shows a circuit consisting of two nonlinear resistors connected in parallel.
Fig. 9.5. Two nonlinear resistors connected in parallel

The characteristics \( v_1-i_1 \) and \( v_2-i_2 \) are depicted in Figs 9.6 and 9.7, respectively.

Since the resistors are connected in parallel, the voltages \( v_1 \) and \( v_2 \) are identical

\[
v = v_1 = v_2 \quad (9.3)
\]
Using KCL at the top node we have

\[ i = i_1 + i_2 . \]  

(9.4)

Thus, the characteristic \( v - i \) can be traced in a graphical manner by adding for each value of \( v \) the corresponding currents \( i_1 \) and \( i_2 \) (see Fig. 9.8).

Fig. 9.8. Graphical construction for finding characteristic \( v - i \)

**Operating points**

Nonlinear circuits driven by DC sources have constant solutions (branch voltages and current), called operating points. The basic question of the analysis of this class of circuits is finding the operating points. If a circuit is simple, the operating point can be found using a graphical approach. To illustrate this approach we consider a typical biasing circuit depicted in Fig. 9.9, including a nonlinear resistor specified by the characteristic shown in Fig. 9.10.

The linear part of this circuit, consisting of the DC voltage source \( E \) and resistor \( R \), is described by the equation

\[ v_b = E + R i_b . \]  

(9.5)
We transcribe this characteristic in the $i_b - v_b$ plane to the $i_a - v_a$ plane. Since

$$v_b = v_a \quad \text{and} \quad i_b = -i_a,$$

we obtain

$$v_a = E - Ri_a .$$  \hspace{1cm} (9.6)

Equation (9.6) describes a straight line, shown in Fig. 9.11. On the same plane we plot the characteristic

$$v_a = f(i_a) .$$

The point of intersection $\left( \hat{i}_a, \hat{v}_a \right)$ is the solution of the set of equations

$$v_a = f(i_a) ,$$

$$v_a = E - Ri_a ,$$

hence, the intersection is the operating point of the circuit shown in Fig. 9.9.
Fig. 9.11. Graphical construction for finding the operation point
Controlled sources are circuit elements very useful in the modeling of electronic devices. A controlled source is a two-port consisting of two branches. A primary branch is either a short circuit or an open circuit. A secondary branch is either a voltage source or a current source. The source waveform depends on a voltage or a current of the primary branch. Thus, there are four types of controlled sources. The controlled sources can be classified as linear or nonlinear. All the controlled sources are shown in Figs 10.1 – 10.4.

Fig. 10.1. Current-controlled voltage sources (CCVS) a) linear, b) nonlinear

Fig. 10.2. Voltage-controlled current sources a) linear, b) nonlinear
Controlled sources are very useful in modeling electronic devices. For example the Ebers-Moll model of an npn bipolar transistor contains two linear current controlled current sources, as illustrated in Fig. 10.5.

Fig. 10.5. The Ebers-Moll model of an npn bipolar transistor
A capacitor is a two-terminal element which stores an electric charge. The simplest example of a capacitor is shown in Fig. 11.1. It is made of two flat parallel metal plates in free space.

When a current $i(t)$ is applied, then a charge $q(t) = Cv(t)$ is induced on the upper plate and an equal but opposite charge is induced on the lower plate. The constant of proportionality, called capacitance, is given approximately by

$$C = \frac{\varepsilon_0 A}{d},$$
where

\[ \varepsilon_0 = \frac{1}{36\pi} \times 10^{-9} \frac{F}{m} \]

is the dielectric constant called permittivity, \( A \) is the plate area and \( d \) is the separation of the plates. The units of capacitance are farads, abbreviated to F.

Equation

\[ q = Cv \quad (11.1) \]

defines the \( v-q \) characteristic of the capacitor. The characteristic is a straight line through the origin with a slope \( C \), as shown in Fig. 11.2.

![Fig. 11.2. Characteristic \( v-q \) of a linear capacitor](image)

A capacitor whose characteristic is a straight line through the origin is called a linear capacitor. Otherwise, the capacitor is said to be nonlinear. An example \( v-q \) characteristic of a nonlinear capacitor is shown in Fig 11.3.
The symbols of a linear and a nonlinear capacitor and the reference directions of $i(t)$ and $v(t)$ are shown in Figs 11.4 and 11.5. In these figures $q$ is the charge on this plate which is pointed by the reference arrow of the current $i$.

Fig. 11.3. Characteristic $v$-$q$ of a nonlinear capacitor

Fig. 11.4. Symbol of a linear capacitor

Fig. 11.5. Symbol of a nonlinear capacitor
The current $i(t)$ is given by the equation

$$i(t) = \frac{dq(t)}{dt}.$$  \hspace{1cm} (11.2)

Using equation (11.1) and (11.2) we obtain

$$i = \frac{dq}{dt} = \frac{d(Cv)}{dt} = C \frac{dv}{dt}.$$  \hspace{1cm} (11.3)

Equation (11.3) expresses capacitor current in terms of capacitor voltage. To express capacitor voltage in terms of capacitor current we replace the variable $t$ with $\tau$ obtaining

$$i(\tau) = C \frac{dv(\tau)}{d\tau}.$$  \hspace{1cm} (11.4)

Next we integrate both sides of (11.4) between $0$ and $t$

$$\int_{0}^{t} i(\tau)d\tau = C \int_{0}^{v(t)} \frac{dv(\tau)}{d\tau}d\tau = C \int_{v(0)}^{v(t)} dv = C(v(t) - v(0)).$$

Hence, it holds

$$v(t) = v(0) + \frac{1}{C} \int_{0}^{t} i(\tau)d\tau.$$  \hspace{1cm} (11.5)

### 11.2 Continuity property

Let us replace $t$ in equation (11.5) by $t + dt$

$$v(t + dt) = v(0) + \frac{1}{C} \int_{0}^{t + dt} i(\tau)d\tau.$$  \hspace{1cm} (11.6)

and assume that $i(t)$ is bounded for all $t$, that is there exists a constant $L$, such that $|i(t)| < L$ for all $t$. 

Notes
Subtracting equation (11.5) from (11.6) we have

$$v(t + dt) - v(t) = \frac{1}{C} \int_0^{t+dt} i(\tau)d\tau .$$

(11.7)

As \( dt \to 0 \), then \( \int_0^{t} i(\tau)d\tau \to 0 \), hence, \( v(t + dt) \to v(t) \).

Thus, voltage across any linear capacitor is a continuous function of time. It means that this voltage cannot jump instantaneously from one value to another.

### 11.3 Energy stored in a capacitor

Consider a capacitor supplied with a generator as shown in Fig. 11.6.

![Fig. 11.6. A capacitor supplied with a generator](image)

The energy delivered by the generator to the capacitor from time \( t_0 \) to \( t \) is given by the equation

$$A(t_0 , t) = \int_{t_0}^{t} p(\tau)d\tau = \int_{t_0}^{t} v(\tau)i(\tau)d\tau ,$$

(11.8)

where \( p \) is the instantaneous power entering the capacitor.
Let \( v(t_0) = 0 \), hence, no charge is stored in the capacitor. We choose this state as the state corresponding to zero energy, i.e. \( w(t_0) = 0 \), where at \( t = t_0 \) is the initial energy of the capacitor. Let us replace \( t \) by another variable \( \tau \) in equation (11.3)

\[
i(\tau) = C \frac{dv(\tau)}{d\tau}
\]

and rewrite it in the form

\[
i(\tau) d\tau = Cdv . \tag{11.9}
\]

Substituting (11.9) into (11.8) yields

\[
A(t_0, t) = \int_{t_0}^{t} v(\tau) i(\tau) d\tau = \int_{0}^{v(t)} vCdv = C \frac{1}{2} v^2 \bigg|_{0}^{v(t)} = \frac{1}{2} C(v(t))^2 . \tag{11.10}
\]

A capacitor is an element that stores energy, but does not dissipate it. Hence, the energy stored in the capacitor at time \( t \) is given by the equation

\[
w(t) = w(t_0) + A(t_0, t) = A(t_0, t) . \tag{11.11}
\]

Since \( w(t_0) = 0 \) and \( A(t_0, t) \) is specified by (11.10), we have

\[
w(t) = \frac{1}{2} Cv^2(t) . \tag{11.12}
\]

### 11.4 Series connection of capacitors

Consider two capacitors connected in series as shown in Fig. 11.7.

![Diagram of two capacitors connected in series](Image)

**Fig. 11.7. Two capacitors connected in series**
Since the same current traverses both capacitors we write, on the basis of (11.5)

\[ v_1(t) = v_1(0) + \frac{1}{C_1} \int_0^t i(\tau) \, d\tau , \quad (11.13) \]

\[ v_2(t) = v_2(0) + \frac{1}{C_2} \int_0^t i(\tau) \, d\tau . \quad (11.14) \]

Using KVL yields

\[ v(t) = v_1(t) + v_2(t) . \quad (11.15) \]

At \( t = 0 \) equation (11.15) becomes

\[ v(0) = v_1(0) + v_2(0) . \quad (11.16) \]

We add equations (11.13) and (11.14) together and apply (11.16)

\[ v(t) = v(0) + \left[ \frac{1}{C_1} + \frac{1}{C_2} \right] \int_0^t i(\tau) \, d\tau . \quad (11.17) \]

Let

\[ \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{C} , \quad (11.18) \]

then

\[ v(t) = v(0) + \frac{1}{C} \int_0^t i(\tau) \, d\tau . \quad (11.19) \]

Equation (11.19) describes the equivalent capacitor of two capacitors connected in series. The initial voltage of this capacitor is specified by (11.16), whereas the capacitance is given by (11.18).
11.5 Parallel connection of capacitors

Figure 11.8 shows two capacitors connected in parallel. Since voltages across the capacitors are identical, both capacitors have the same initial voltage $v(0)$.

\[ v(t) \]

\[ i(t) \quad i_1(t) \quad i_2(t) \]

\[ C_1 \quad C_2 \]

Fig. 11.8. Two capacitors connected in parallel

Currents flowing through the capacitors are given by the equations

\[ i_1(t) = C_1 \frac{dv(t)}{dt}, \quad i_2(t) = C_2 \frac{dv(t)}{dt}. \]

Applying KCL at the top node and the above equations yields

\[ i(t) = i_1(t) + i_2(t) = (C_1 + C_2) \frac{dv(t)}{dt} = C \frac{dv(t)}{dt}, \]

where

\[ C = C_1 + C_2. \]  \hspace{1cm} (11.20)

Formula (11.20) gives the capacitance of the equivalent capacitor.
12. Inductor

12.1 Introduction

Figure 12.1 shows an inductor made of wire wound around a core, made of a nonmetallic material.

![Fig. 12.1. An example inductor](image)

When the device is supplied with a time varying current source, a magnetic flux is induced and circulates inside the core. The magnetic flux linkage \( \phi(t) \), being the total flux linked by all turns of the coil, is proportional to the current

\[
\phi(t) = L i(t) ,
\]

(12.1)

where the coefficient \( L \) is called inductance. The units of magnetic flux are webers (W), the units of inductance are henrys (H).
If the core shown in Fig. 12.1 is a toroid made of material having the magnetic constant (permeability) $\mu_0$, then the inductance is given by the formula

$$L = \mu_0 \frac{N^2 A}{l},$$

(12.2)

where $\mu_0 = 4\pi \cdot 10^{-7} \text{H/m}$, $N$ is the number of turns, $A$ is the cross-sectional area of the core, $l$ is the midcircumference along the core.

The equation

$$\phi = Li$$

(12.3)

defines $i - \phi$ characteristic of the inductor. The characteristic is a straight line through the origin with a slope equal to $L$ (see Fig. 12.2). In such a case the inductor is classified as linear.

Otherwise, if the characteristic is not a straight line through the origin, the inductor is called nonlinear. The typical characteristic of a nonlinear inductor is shown in Fig. 12.3.
The symbols of linear and nonlinear inductors are shown in Fig. 12.4.

According to Faraday’s induction law

\[ v(t) = \frac{d\phi(t)}{dt} . \]  

(12.4)
Substituting (12.3) into (12.4) yields

\[ v = \frac{d}{dt} (Li) = L \frac{di}{dt} . \]  

(12.5)

To express the current \( i \) flowing through a linear inductor in terms of the voltage \( v \) across the inductor we replace \( t \) with \( \tau \) and rewrite equation (12.5) in the form

\[ v(\tau) = L \frac{d(i(\tau))}{d\tau} . \]  

(12.6)

Next we integrate both sides of equation (12.6) between 0 and \( t \)

\[ \int_{0}^{t} v(\tau) d\tau = L \int_{0}^{t} \frac{d(i(\tau))}{d\tau} d\tau = L \int_{i(0)}^{i(t)} di = L(i(t) - i(0)) . \]

Hence, we have

\[ i(t) = i(0) + \frac{1}{L} \int_{0}^{t} v(\tau) d\tau . \]  

(12.7)

### 12.2 Continuity property

Let us replace \( t \) in equation (12.7) by \( t + dt \)

\[ i(t + dt) = i(0) + \frac{1}{L} \int_{0}^{t+dt} v(\tau) d\tau \]  

(12.8)

and assume that \( v(t) \) is bounded for all \( t \), that is, there exists a constant \( M \) such that \( |v(t)| < M \) for all \( t \). Subtracting equation (12.7) from (12.8) yields

\[ i(t + dt) - i(t) = \frac{1}{L} \left( \int_{0}^{t+dt} v(\tau) d\tau - \int_{0}^{t} v(\tau) d\tau \right) = \frac{1}{L} \int_{t}^{t+dt} v(\tau) d\tau . \]
Since \( \frac{1}{L} \int_{t}^{t+dt} v(\tau) d\tau \to 0 \) as \( dt \to 0 \), then \( i(t + dt) \to i(t) \). Thus, current flowing through any linear inductor is a continuous function of time. This means that inductor current cannot jump instantaneously from one value to another.

### 12.3 Hysteresis

Ferromagnetic core inductors exhibit the hysteresis phenomenon as depicted in Fig. 12.5.

![Fig. 12.5. Hysteresis phenomenon](image)

The characteristic shown in Fig. 12.5 is obtained by increasing the current \( i \) from 0 to \( i_1 \), next decreasing this current from \( i_1 \) to \( i_3 \) and after that increasing from \( i_3 \) to \( i_1 \). Thus, the flux decreases according to the upper branch 2 and increases according to the lower branch 3. As a result, a close curve is traced. The magnetic flux becomes zero for the negative value \( i_2 \) and the positive value \( i_4 \) of the current.
12.4 Energy stored in an inductor

Consider an inductor supplied with a generator as shown in Fig. 12.6.

![Diagram of inductor supplied with a generator](image)

Fig. 12.6. An inductor supplied with a generator

The energy delivered by the generator to the inductor from time \( t_0 \) to \( t \) is given by the equation

\[
A(t_0, t) = \int_{t_0}^{t} p(\tau) d\tau = \int_{t_0}^{t} v(\tau) i(\tau) d\tau , \tag{12.9}
\]

where \( p \) is the instantaneous power entering the inductor. Let \( i(t_0) = 0 \), consequently also the magnetic flux equals zero and no magnetic field exists in the inductor. Such a state can be considered a state corresponding to zero energy stored. Let us substitute (12.6) into (12.9). Then, we have

\[
A(t_0, t) = \int_{t_0}^{t} L i(\tau) \frac{di(\tau)}{d\tau} d\tau = L \int_{i(t_0)}^{i(t)} i d\tau = \frac{1}{2} L \frac{d^2}{di(t_0)} = \frac{1}{2} L \left( i^2(t) - i^2(t_0) \right). \]

Since \( i(t_0) = 0 \) the energy initially stored in the inductor is also equal to zero, \( w(t_0) = 0 \) and the energy delivered to the inductor from \( t = t_0 \) to \( t \) is

\[
A(t_0, t) = \frac{1}{2} L i^2(t). \tag{12.10}
\]
An inductor is an element that stores energy, but does not dissipate it. Hence, the energy stored in the inductor at time $t$ is given by the equation

$$w(t)= w(t_0) + A(t_0, t) = \frac{1}{2} Li^2(t). \quad (12.11)$$

### 12.5 Series connection of inductors

Consider two inductors connected in series as shown in Fig. 12.7.

![Fig. 12.7. Two inductors connected in series](image)

Since identical currents flow through the inductors, both inductors have the same initial current $i(0)$.

KVL leads to the equation

$$v(t) = v_1(t) + v_2(t),$$

where

$$v_1(t) = L_1 \frac{di(t)}{dt}, \quad v_2(t) = L_2 \frac{di(t)}{dt}.$$  

Hence, we have

$$v(t) = L_1 \frac{di(t)}{dt} + L_2 \frac{di(t)}{dt} = (L_1 + L_2) \frac{di(t)}{dt} = L \frac{di(t)}{dt},$$

where

$$L = L_1 + L_2. \quad (12.12)$$

Formula (12.12) gives the inductance of the equivalent inductor.
12.6 Parallel connection of inductors

Figure 12.13 shows two inductors connected in parallel.

We write KCL equation

\[ i(t) = i_1(t) + i_2(t) \]

and substitute into this equation

\[ i_1(t) = i_1(0) + \frac{1}{L_1} \int_0^t v(\tau) \, d\tau \]

and

\[ i_2(t) = i_2(0) + \frac{1}{L_2} \int_0^t v(\tau) \, d\tau . \]

As a result we obtain

\[ i(t) = i(0) + \left( \frac{1}{L} \right) \int_0^t v(\tau) \, d\tau , \]

(12.13)

where

\[ i(0) = i_1(0) + i_2(0) \]

(12.14)
and

\[ \frac{1}{L_1} + \frac{1}{L_2} = \frac{1}{L} \]  

(12.15)

or

\[ L = \frac{L_1L_2}{L_1 + L_2} \]  

(12.16)

Thus, the parallel connection of two inductors can be replaced by a single inductor having the inductance specified by (12.16), with the initial condition (12.14).
13. Operational-amplifier circuits

13.1 Description of the operational amplifier

An operational amplifier is a multi-terminal semiconductor device. Figure 13.1 shows the symbol of this device, including inside triangle interconnected transistors, resistors and a power supply voltage source. The terminals available for external connections are called: an inverting input, a noninverting input, an output, and external ground.

In Fig. 13.2 voltages and currents are introduced, where $i_-$ and $i_+$ denote the currents entering the operation amplifier, $v_d$ is the input voltage, $v_0$ denotes the output voltage, and $i_0$ the output current.

---

Notes

---

78
The operational amplifier is described by the following set of equations:

\[
\begin{align*}
    i_- &= 0, \\
    i_+ &= 0, \\
    v_0 &= f(v_d),
\end{align*}
\]

where \( f(v_d) \) is a function shown in Fig. 13.3. For \( v_d \) belonging to a very small interval \([-\varepsilon, \varepsilon]\) function \( f(v_d) \) is linear \( f(v_d) = A \), where \( A \) equals at least 100 000.

At \( v_0 = \pm E_{sat} \) the function \( f(v_d) \) saturates. Since the slope \( A \) is very large we can assume that \( A = \infty \). As a result we obtain the ideal model of the operational amplifier, having the characteristic \( v_0 = f(v_d) \) shown in Fig. 13.4.
The characteristic is piecewise-linear and consists of three segments. They define three operating regions called: a linear region, a +saturation region, and a –saturation region.

\[ v_0 = f(v_d) \]

Fig. 13.4. The characteristic of the ideal model of the operational amplifier

The linear region is described by the following equations:

\[ i_- = 0 , \]
\[ i_+ = 0 , \]
\[ v_d = 0 . \]  \hspace{1cm} (13.1)

The output voltage in this region satisfies the inequality

\[ -E_{sat} < v_0 < E_{sat} \]

called the validating inequality.
In the +saturation region the operational amplifier is described by the equations

\[ i_- = 0, \]
\[ i_+ = 0, \] (13.3)
\[ v_0 = E_{sat}, \]

and the output voltage satisfies the validating inequality

\[ v_d > 0. \] (13.4)

The above relationships lead to the model of the operational amplifier operating in the +saturation region, as shown in Fig. 13.5.

[Diagram of the operational amplifier in +saturation region]

Fig. 13.5. The model of the operational amplifier operating in the +saturation region

In the –saturation region the operational amplifier is specified by the following set of equations:

\[ i_- = 0, \]
\[ i_+ = 0, \] (13.5)
\[ v_0 = -E_{sat}, \]

whereas the validating inequality is

\[ v_d < 0. \] (13.6)
Hence, we obtain the model of the operational amplifier operating in the –saturation region, depicted in Fig. 13.6.

13.2 Examples

To explain the method for the analysis of the circuits containing operational amplifiers we consider two examples.

Example 13.1
Figure 13.7 shows a circuit called an inverter. We assume that operational amplifier operates in the linear region and express the output voltage \( v_0 \) in terms of the input voltage \( v_i \).

Fig. 13.6. The model of the operational amplifier operating in the –saturation region

Fig. 13.7. The inverter containing the operational amplifier
Since the operation amplifier operates in the linear region, it is described by the set of equations (13.1). Especially $v_d = 0$, hence,

$$v_1 = v_i \text{ and } i_1 = \frac{v_1}{R_1} = \frac{v_i}{R_1}.$$  

KCL applied at node $P$ leads to the equation

$$i_2 = i_1.$$  

Consequently, it holds

$$v_2 = R_2i_2 = R_2i_1 = \frac{R_2}{R_1}v_i.$$  

Using KVL we write

$$v_0 = -v_2.$$  

Thus, the output voltage equals

$$v_0 = -\frac{R_2}{R_1}v_i.$$  

The obtained result is valid if $v_0$ satisfies the validating inequality

$$-E_{sat} < v_0 < E_{sat}.$$  

Substituting (13.7) we obtain

$$-E_{sat} < -\frac{R_2}{R_1}v_i < E_{sat},$$

or

$$-\frac{R_1}{R_2}E_{sat} < v_i < \frac{R_1}{R_2}E_{sat}.$$  

(13.8)
Thus, the output voltage is specified by equation (13.7) if the input voltage is selected so that the inequality (13.8) is satisfied.

**Example 13.2**

Figure 13.8 shows a circuit called a voltage comparator.

![Diagram of a voltage comparator](image)

Fig. 13.8. The voltage comparator containing the operational amplifier

In this circuit the operational amplifier is allowed to operate in all the three regions. We wish to find the transfer characteristic $v_0 = f(v_i)$.

Using KVL we write

$$v_d = v_i - V.$$  \hspace{1cm} (13.9)

Below we consider 3 cases.

1. The operational amplifier operates in the linear region. In this case $v_d = 0$, hence using (13.9) we obtain

$$v_i = V.$$  

This result is correct if the validating inequality $-E_{sat} < v_0 < E_{sat}$ is satisfied. As a result, we find the segment (1) of the transfer characteristic shown in Fig. 13.9.
2. The operational amplifier operates in the +saturation region. Thus, $v_d > 0$, $v_0 = E_{sat}$, and equation (13.9) gives

$$v_i > V.$$  

As a result, we obtain the segment of the transfer characteristic shown in Fig. 13.9.

3. The operational amplifier operates in the –saturation region. Similarly, as in case 2, we write the relations:

$$v_d < 0, \quad v_0 = -E_{sat}, \quad v_i < V,$$

which lead to segment of the characteristic (see Fig. 13.9).

![Fig. 13.9](image)

**Fig. 13.9.** The transfer characteristic $v_0 = f(v_i)$ of the voltage comparator shown in Fig. 13.8

### 13.3 Finite gain model of the operational amplifier

In this section we consider the operational amplifier model specified by the characteristic $v_0 = f(v_d)$ shown in Fig. 13.10. Unlike the ideal model the slope of the segment passing through the origin is very large, but finite.
In the linear region the model is described by the equation
\[ v_0 = A v_d \]
and the validating inequality
\[ -\varepsilon < v_d < \varepsilon . \]
Hence, it follows the equivalent circuit shown in Fig. 13.11.

Fig. 13.10. The characteristic \( v_0 = f(v_d) \) of nonideal model of the operational amplifier

Fig. 13.11. The equivalent circuit of the operational amplifier operating in the linear region
In the +saturation region the mathematical representation of the model is

\[ v_0 = E_{sat}, \]
\[ v_d > \varepsilon, \]

whereas in the –saturation region the model is described by

\[ v_0 = -E_{sat}, \]
\[ v_d < -\varepsilon, \]

The corresponding equivalent circuits are shown in Figs 13.12 and 13.13.

![Fig. 13.12. The equivalent circuit of the operational amplifier operating in the +saturation region](image1)

![Fig. 13.12. The equivalent circuit of the operational amplifier operating in the -saturation region](image2)
14. First order circuits driven by DC sources

Circuits made of one inductor or one capacitor, resistors and sources are called first order dynamic circuits. In this section we study circuits driven by DC sources.

14.1 Resistor-inductor circuits

We consider a simple circuit consisting of a DC voltage source, a resistor and an inductor, with an initial current $i(0) = I_0$ (see Fig. 14.1).

Applying KVL we write

$$0 = -L \frac{dv_L}{dt} - Rv_R - V.$$

Since $v_L = L \frac{di}{dt}$,

we obtain, after simple rearrangements

$$L \frac{di}{dt} + Ri = V, \quad i(0) = I_0. \quad (14.1)$$
Equation (14.1) can be presented in the form

\[ \frac{di}{dt} + \frac{1}{\tau} i = \frac{1}{L} V, \quad (14.2) \]

where

\[ \tau = \frac{L}{R}, \quad (14.3) \]

is called a time constant. We rewrite equation (14.2) in the form

\[ \frac{di}{dt} = \frac{1}{L} V - \frac{1}{\tau} i = \frac{V}{\tau} - \frac{V - i}{\tau} = \frac{R - i}{\tau} \]

and separate the variables \( i \) and \( t \)

\[ \frac{di}{i - \frac{V}{R}} = -\frac{dt}{\tau}. \quad (14.4) \]

Next, we integrate equation (14.4)

\[ \int \frac{di}{i - \frac{V}{R}} = -\frac{1}{\tau} \int dt, \]

finding

\[ \ln \left| i - \frac{V}{R} \right| = -\frac{t}{\tau} + K, \]

where the constant \( K \) may be written as \( K = \ln|A| \), where \( A \) is another constant.
Hence, we have

\[ \ln \left( \frac{i - V}{R} \right) = -\frac{t}{\tau}, \]

or

\[ i = \frac{V}{R} + Ae^{-\frac{t}{\tau}}. \]

(14.5)

To determine \( A \) we write equation (14.5) at \( t = 0 \) and set \( i(0) = I_0 \)

\[ I_0 = \frac{V}{R} + A. \]

(14.6)

Substituting \( A \), given by (14.6) into equation (14.5) yields

\[ i(t) = \frac{V}{R} + \left( I_0 - \frac{V}{R} \right) e^{-\frac{t}{\tau}}. \]

(14.7)

When \( t = 0 \), we have \( i(0) = I_0 \), which is the correct initial conditions. When \( t \to \infty \), the second term on the right side of equation (14.7) vanishes, hence, it holds \( i(\infty) = \frac{V}{R} \).

Thus, we have

\[ i(t) = i(\infty) + (i(0) - i(\infty)) e^{-\frac{t}{\tau}}, \]

(14.8)

where \( i(0) \) is the initial value of \( i(t) \), whereas \( i(\infty) \) is the steady-state value of \( i(t) \).

Since the steady-state current is constant, the voltage across the inductor

\[ v_L = L \frac{di}{dt} \]

equals zero.

Notes
Thus, the inductor behaves like a short circuit and the circuit considered in the steady-state (as $t \to \infty$) consists of the voltage source $V$ and the resistor $R$ (see Fig. 14.2).

![Circuit Diagram](image)

**Fig. 14.2. Model of the circuit shown in Fig. 14.1 in the steady-state**

The plot of $i(t)$ is shown in Fig. 14.3.

![Plot Diagram](image)

**Fig. 14.3. Plot of $i(t)$ and graphical interpretation of the time constant**

The initial rate of changing of the current is given by

$$\left. \frac{di}{dt} \right|_{t=0} = \tan \alpha = -\frac{1}{\tau} (i(0) - i(\infty)) e^{-\frac{t}{\tau}} \bigg|_{t=0} = -\frac{1}{\tau} (i(0) - i(\infty)).$$

(14.9)
On the other hand

\[
\frac{df}{dt}_{t=0} = \tan \beta = \frac{-\left(i(0) - i(\infty)\right)}{x}.
\]  

(14.10)

Equalizing (14.9) and (14.10) we find

\[x = \tau\,.
\]

Thus, to find graphically the time constant, it is necessary to draw the tangent of the current at \(t = 0\) and determine its intercept with the horizontal line passing through the point \((0, i(\infty))\).

In a very special case, when \(V = 0\), \(i(\infty) = 0\) and equation (14.8) becomes

\[i(t) = I_0 e^{-\frac{t}{\tau}}.
\]

(14.11)

The plot of \(i(t)\) is shown in Fig. 14.4 and the time constant \(\tau = x\).

Fig. 14.4. Plot of \(i(t)\) in the circuit with \(V=0\)

At \(t = \tau\), \(i(\tau) = I_0 e^{-\frac{\tau}{\tau}} = I_0 e^{-1} = 0.368 I_0\). Thus, in one time constant the current has declined to 0.368 of its initial value (see Fig. 14.4).
At $t = 5\tau$, $i(5\tau) = I_0 e^{-5} = 0.0067 I_0$ and the current is a negligible fraction of its initial value. Therefore, we usually assume that after 5 time constants the current is in the steady state.

**Example 14.1**
The switch in the circuit shown in Fig. 14.5 is open until the steady state prevails and then it is closed. Assuming that the closing occurs at $t = 0$, we find the current $i_L(t)$.

![Fig. 14.5. A first order circuit driven by a DC voltage source](image)

When the switch is open, the circuit is in the steady state. The current is constant, hence, the voltage across the inductor, given by $v_L = L \frac{di}{dt}$, is equal to zero. In such a case, the inductor can be replaced by a short circuit and the circuit consists of the DC voltage source and resistors $R_1$, $R_2$ connected in series. At the left hand side of $t = 0$, labeled $0^-$, the switch is open and

$$i_L(0^-) = \frac{V}{R_1 + R_2}.$$  \hspace{1cm} (14.12)

At $t = 0$ the switch is closed. However, the current $i_L(t)$ flowing through the inductor remains unchanged at this moment, due to the continuity property

$$i_L(0) = i_L(0^-) = \frac{V}{R_1 + R_2}.$$  \hspace{1cm} (14.12)
When the switch is closed the circuit has the same form as the circuit shown in Fig. 14.1. Hence, the current $i_L(t)$ is given by equation (14.8), repeated below

$$i(t) = i(\infty) + (i(0) - i(\infty))e^{-\frac{t}{\tau}}.$$  (14.13)

At $t = \infty$ the circuit is in the steady state, voltage across the inductor equals zero and the current is given by the formula

$$i_L(\infty) = \frac{V}{R_2}.$$  (14.14)

Substituting (14.12) and (14.14) into equation (14.13) yields

$$i_L(t) = \frac{V}{R_2} + \left(\frac{V}{R_1 + R_2} - \frac{V}{R_2}\right)e^{-\frac{t}{\tau}},$$  (14.15)

where

$$\tau = \frac{L}{R_2}.$$  (14.16)

The plot $i_L$ against $t$ is shown in Fig. 14.6.

![Graph](image-url)  

Fig. 14.6. Plot $i_L(t)$ in the circuit shown in Fig. 14.5
Now we consider a general class of first order circuits consisting of DC sources, resistors, and one inductor. We extract the inductor from the circuit, obtaining the circuit shown in Fig. 14.7.

![Circuit with the extracted inductor](image1)

Fig. 14.7. Circuit with the extracted inductor

Then, we apply the Thevenin theorem to reduce the circuit to the form shown in Fig. 14.8.

![Thevenin circuit](image2)

Fig. 14.8. The Thevenin circuit of the resistive one-port shown in Fig. 14.7

In this circuit the current is given by equation (14.8) repeated below

95
where

\[ i_L(t) = i_L(\infty) + (i_L(0) - i_L(\infty))e^{-\frac{t}{\tau}}, \]

\[ i_L(\infty) = \frac{V_0}{R_{eq}} \quad \text{and} \quad \tau = \frac{L}{R_{eq}}. \]

14.2 Resistor-capacitor circuits

Another first order dynamic circuit is an \( RC \) circuit consisting of one capacitor, resistors, and sources. In this section we study circuits driven by DC sources.

Let us consider a simple \( RC \) circuit, driven by a DC voltage source, as shown in Fig. 14.9.

To describe the circuit we write KVL equation

\[ v_R + v_C = V, \]

where

\[ v_R = Ri_C \]

and

\[ i_C = C \frac{dv_C}{dt}. \]
Hence, we have

\[ RC \frac{dv_C}{dt} + v_C = V , \]  

(14.17)

We divide both sides of equation (14.17) by RC

\[ \frac{dv_C}{dt} + \frac{1}{RC} v_C = \frac{1}{RC} V \]

and define the time constant

\[ \tau = RC . \]  

(14.18)

As a result, we obtain the equation

\[ \frac{dv_C}{dt} + \frac{1}{\tau} v_C = \frac{1}{\tau} V , \]  

(14.19)

with the initial condition

\[ v_C(0) = V_0 . \]

Thus, the circuit shown in Fig. 14.9 is described by equation (14.19). From a mathematical point of view this equation is similar to equation (14.2), therefore, its solution is as follows

\[ v_C(t) = v_C(\infty) + (v_C(0) - v_C(\infty))e^{-\frac{t}{\tau}} , \]  

(14.20)

where \( v_C(\infty) \) is the steady state solution, at \( t = \infty \). Thus, the voltage \( v_C \) at the steady state is constant, its derivative equals zero. This means that no current flows through the capacitor in the steady state. Consequently, the capacitor can be removed and the circuit, in the steady state, becomes as shown in Fig. 14.10.

Using KVL in the circuit depicted in Fig. 14.2 we find

\[ v_C(\infty) = V . \]
Figures 14.11 and 14.12 show the plot $v_C(t)$ in two cases: when $V > v_C(0)$ and when $V < v_C(0)$. The graphical interpretation of the time constant is also shown in these figures.
The current $i_C(t)$ is given by

$$i_C = C \frac{dv_C}{dt} = C(v_C(0) - v_C(\infty)) \left( -\frac{1}{\tau} \right) e^{-\frac{t}{\tau}} = \frac{v_C(\infty) - v_C(0)}{R} e^{-\frac{t}{\tau}}.$$ 

Its plot is shown in Fig. 14.13.

![Plot of current $i_C$ against time](image)

Fig. 14.13. Plot of current $i_C$ against time

Alternatively, the current can be found as follows. First we compute the voltage $v_R$ (see Fig. 14.9) using KVL

$$v_R(t) = V - v_C(t) = V - \left( V + (v_C(0) - V)e^{-\frac{t}{\tau}} \right) = (V - v_C(0))e^{-\frac{t}{\tau}}.$$ 

Next, we apply Ohm’s law

$$i_C = \frac{v_R}{R} = \frac{V - v_C(0)}{R} e^{-\frac{t}{\tau}}.$$ 

Now we consider a circuit consisting of DC sources, resistors, and one capacitor. We extract the capacitor from the circuit, obtaining the circuit shown in Fig. 14.14.
Next, we apply Thevenin’s theorem to reduce the circuit to the form shown in Fig. 14.15.

In this circuit the voltage $v_C(t)$ is given by equation (14.20) repeated below

$$v_C(t) = v_C(\infty) + (v_C(0) - v_C(\infty))e^{-\frac{t}{\tau}} \tag{14.21}$$

where

$$v_C(\infty) = V_{0C} \quad \text{and} \quad \tau = R_{eq}C .$$
Example 14.2

In the circuit shown in Fig. 14.16 the switch has been open for a long time prior to $t = 0$ when it is closed. Find the voltage $v_C(t)$ for $t \geq 0$.

![Circuit Diagram](image)

Since at $t = 0^-$, the circuit is in the steady state, no current flows through the capacitor and the circuit becomes as shown in Fig. 14.17.

![Model Diagram](image)

In the circuit of Fig. 14.17 it holds

$$v_C(0^-) = V_S.$$
Since the capacitor satisfies the continuity property

\[ v_C(0) = v_C(0^-) = V_S. \]

When the switch is closed the circuit has the form shown in Fig. 14.18.

![Circuit Diagram](image)

Fig. 14.18 The circuit of Fig. 14.16 when the switch is closed

It consists of a resistive one-port terminated by the capacitor \( C \). Using Thevenin’s theorem we obtain the equivalent circuit shown in Fig. 14.19, where

\[
R_{eq} = R_3 + \frac{R_1R_2}{R_1 + R_2} = 10 \Omega,
\]

\[
V_{OC} = \frac{V_S}{R_1 + R_2}R_2 = 6 \text{ V}.
\]

![Equivalent Circuit Diagram](image)

Fig. 14.19. Circuit equivalent to the circuit of Fig. 14.18
In the circuit shown in Fig. 14.19 the voltage $v_C(t)$ is given by equation (14.21) repeated below

$$v_C(t) = v_C(\infty) + (v_C(0) - v_C(\infty))e^{-\frac{t}{\tau}},$$

where

$$\tau = R_{eq}C = 10^{-5} \text{ S}.$$  

In the steady state the voltage $v_C$ is constant, hence, $i_C = C\frac{dv_C}{dt} = 0$ and $v_C(\infty) = V_{0C} = 6 \text{ V}$. Having $v_C(\infty) = 6 \text{ V}$, $v_C(0) = V_S = 12 \text{ V}$, $\tau = 10^{-5} \text{ S}$, we obtain

$$v_C(t) = \left(6 + 6e^{-\frac{t}{10^{-5}}} \right).$$

The plot of $v_C(t)$ is shown in Fig. 14.20.

![Plot of $v_C(t)$ in the circuit shown in Fig. 14.17](image)
15. Sinusoidal steady-state analysis

15.1 Preliminary discussion

The proceeding discussion was focused on the analysis of circuits driven by constant sources. However, many signals encountered in practice are time-varying. In power systems the sinusoidal signal is the most popular. In this section we concentrate on the analysis of linear circuits driven by sinusoidal sources.

Let us consider a sinusoidal signal

\[ f(t) = A \cos \omega t \],

(15.1)

where \( A \) is called an amplitude and \( \omega \) is called an angular frequency measured in radians/second (abbreviated to rad/s). The signal (15.1) repeats itself after every \( T \) seconds, where \( T \) is called a period. The product of \( \omega \) and \( t \) is an angle in radians. The angle corresponding to one period is \( 2\pi \) radians. Therefore, we may write

\[ 2\pi = \omega T \]

or

\[ T = \frac{2\pi}{\omega} \].

(15.2)

The number of periods completed each second is known as the frequency \( f \) of the signal. Thus, the frequency is given by

\[ f = \frac{1}{T} \].

The unit of frequency is hertz (abbreviated to Hz). Using equation (15.2) we have

\[ \omega = \frac{2\pi}{T} = 2\pi f \].
The waveform of the signal (15.1) is shown in Fig. 15.1. In this case, the positive peak of the signal occurs at $t = 0$.

A general waveform of sinusoidal signal is shown in Fig. 15.2, where the positive peak occurs at $t = t_1$. 

---

**Notes**
This signal can be expressed as follows
\[ f(t) = A \cos(\omega(t - t_i)) = A \cos(\omega t - \omega t_i) = A \cos(\omega t + \phi) \]  
(15.3)

where
\[ \phi = -\omega t_i = -\frac{2\pi}{T}t_i \]  
(15.4)
is called an initial phase or simply a phase.

**Example 15.1**

Let us consider a sinusoidal voltage, shown in Fig. 15.3.

![Fig. 15.3. An example of a sinusoidal voltage](image)

The quantities characterising this signal are as follows:

\[ A = 5 \text{ V}, \]
\[ T = 10 \text{ ms}, \]
\[ f = \frac{1}{10 \cdot 10^{-3}} = 100 \text{Hz}, \]
\[ \omega = 2\pi f = 2\pi \cdot 100 = \frac{628}{s}, \]
\[ \phi = -\frac{2\pi}{T}t_1 = -\frac{2\pi}{10}1.5 = -0.3\pi \text{ radians}. \]

In practice however, we use degrees for the phase, because we have a physical feel for angles measured in degrees.

Since \( \pi \rightarrow 180^\circ \), then \( -0.3\pi \rightarrow -0.3 \cdot 180^\circ = -54^\circ \). Using the obtained results we may write

\[ v(t) = 5\cos(628t - 54^\circ). \]

Note that for any integer \( n \) it holds

\[ A\cos(\omega t + \phi) = A\cos(\omega t + \phi + 2\pi n). \]

Hence, there are infinitely many phases specified by \( \phi + 2\pi n \).

To determine the phase uniquely we chose the positive peak of the sinusoidal signal closed to the origin, as in Example 15.1.

Consider two sinusoidal signals at the same frequency

\[ a(t) = A\cos(\omega t + \phi_a) \]

and

\[ b(t) = B\cos(\omega t + \phi_b). \]

If \( \phi_a \neq \phi_b \), then the phase difference between them is defined as

\[ \phi_a - \phi_b. \]

If \( \phi_a > \phi_b \), then \( a(t) \) is said to lead \( b(t) \) by \( (\phi_a - \phi_b) \). If \( \phi_a < \phi_b \), then \( a(t) \) is said to lag behind \( b(t) \) by \( (\phi_b - \phi_a) \).
Example 15.2
We calculate the phase difference between the current
\[ i(t) = 20\cos(700t + 10^\circ) \text{A} \]
and the voltage
\[ v(t) = 100\sin(700t + 50^\circ) \text{V} \].
We cannot do it directly, because the current and the voltage have different forms, one is cosine whereas the other is sine. Therefore, we first convert \( v(t) \) to a cosine expression using the trigonometric identity
\[ \sin \alpha = \cos(\alpha - 90^\circ) \].
As a result, we obtain
\[ v(t) = 100\cos(700t + 50^\circ - 90^\circ) = 100\cos(700t - 40^\circ) \text{V} \].
Hence, the phase difference between \( v(t) \) and \( i(t) \) is
\[ \phi_v - \phi_i = -40^\circ - 10^\circ = -50^\circ \].
This means that \( v(t) \) lags behind \( i(t) \) by \( 50^\circ \).

15.2 Phasor concept

We will study linear circuits, driven by sinusoidal sources at the same frequency, in the steady-state. In such a case, all branch currents and voltages have sinusoidal waveforms. The sinusoidal steady-state analysis is called AC analysis. An efficient approach to the AC analysis uses the concept of phasors.

Let us consider a sinusoidal signal
\[ A_m\cos(\omega t + \phi) \].
To his signal we associate a complex number \( A \) called a phasor, according to the rule
Thus, for the given sinusoid the phasor is specified uniquely. Conversely, knowing the frequency $\omega$, the phasor $A$ specifies uniquely the sinusoid by the formula

$$A = A_m e^{j\phi}. \quad (15.6)$$

To illustrate graphically the relation between sinusoidal function (15.5) and associated phasor (15.6) we consider the expression

$$A e^{j\omega t} = A_m e^{j(\omega t + \phi)}. \quad (15.8)$$

In the complex plane this expression describes a vector rotating counter-clockwise at the angular velocity of $\omega$, as shown in Fig. 15.4.

At $t = 0$, we have $A e^{j\omega t} = A = A_m e^{j\phi}$. This is a complex number that we may plot as a vector. For any $t > 0$ expression (15.8) represents a vector having the same length $|A| = |A_m| = A_m$ and the phase equal to $\omega t + \phi$. Thus, the phase of the vector increases, which means that it rotates counter-clockwise.
Next we project orthogonally the tip of the vector on the horizontal axis and label it $x(t)$. Hence, according to Euler’s formula, we obtain

$$A_m e^{j(\omega t + \phi)} = A_m \cos(\omega t + \phi) + j A_m \sin(\omega t + \phi),$$

$$x(t) = A_m \cos(\omega t + \phi).$$

Thus, sinusoidal function (15.5) equals the orthogonal projection of the phasor rotating counter-clockwise at the angular velocity of $\omega$.

15.3 Phasor formulation of circuit equations

Let us consider a linear circuit driven by sinusoidal sources, all at the same frequency $\omega$. In the steady-state all branch voltages and currents in this circuit are sinusoidal at the frequency $\omega$. Figure (15.5) shows an arbitrary node of the circuit.

The sinusoidal currents flowing through the branches meeting at this node are labeled as follows

$$i_1(t) = I_{m_1} \cos(\omega t + \phi_{i_1}), \quad i_2(t) = I_{m_2} \cos(\omega t + \phi_{i_2}),$$

$$i_3(t) = I_{m_3} \cos(\omega t + \phi_{i_3}), \quad i_4(t) = I_{m_4} \cos(\omega t + \phi_{i_4}).$$
KVL applied at the node gives
\[ i_1(t) - i_2(t) - i_3(t) + i_4(t) = 0 \] \hspace{1cm} (15.9)

Now, we write this equation in terms of the phasors
\[ I_1 - I_2 - I_3 + I_4 = 0 \] \hspace{1cm} (15.10)

Below it is proved that equation (15.10) implies equation (15.9).

We multiply both sides of equation (15.10) by \( e^{j\omega t} \)
\[ I_1 e^{j\omega t} - I_2 e^{j\omega t} - I_3 e^{j\omega t} + I_4 e^{j\omega t} = 0 \] \hspace{1cm} (15.11)
and create the phasors of the sinusoidal currents \( i_1(t), \ldots, i_4(t) \):
\[ I_1 = I_{m_1} e^{j\phi_1}, \quad I_2 = I_{m_2} e^{j\phi_2}, \quad I_3 = I_{m_3} e^{j\phi_3}, \quad I_4 = I_{m_4} e^{j\phi_4}. \]

Hence, it holds
\[ I_{m_1} e^{j(\omega t + \phi_1)} - I_{m_2} e^{j(\omega t + \phi_2)} - I_{m_3} e^{j(\omega t + \phi_3)} + I_{m_4} e^{j(\omega t + \phi_4)} = 0. \] \hspace{1cm} (15.12)

Next, we apply Euler’s formula \( e^{j\alpha} = \cos \alpha + j\sin \alpha \) to each term of equation (15.12), finding after simple rearrangements
\[ I_{m_1} \cos(\omega t + \phi_1) - I_{m_2} \cos(\omega t + \phi_2) - I_{m_3} \cos(\omega t + \phi_3) + I_{m_4} \cos(\omega t + \phi_4) + \] \[ + \left[ I_{m_1} \sin(\omega t + \phi_1) - I_{m_2} \sin(\omega t + \phi_2) - I_{m_3} \sin(\omega t + \phi_3) + I_{m_4} \sin(\omega t + \phi_4) \right] = 0. \] \hspace{1cm} (15.13)

Equation (15.13) holds if both the real and the imaginary part is equal to zero. Hence, we may write
\[ I_{m_1} \cos(\omega t + \phi_1) - I_{m_2} \cos(\omega t + \phi_2) - I_{m_3} \cos(\omega t + \phi_3) + I_{m_4} \cos(\omega t + \phi_4) = 0. \] \hspace{1cm} (15.14)

We recognize the terms of equation (15.14) as the sinusoidal currents \( i_1(t), \ldots, i_4(t) \) and rewrite this equation in the form
\[ i_1(t) - i_2(t) - i_3(t) + i_4(t) = 0, \]
being KCL equation (15.9). In this way, we have proved that equation (15.10) implies equation (15.9). Thus, in the sinusoidal steady-state KCL equations can be written down directly in terms of phasors, replacing any sinusoid by its phasor.
Similarly, we prove that KVL equations can be written down directly in terms of phasors.

**Branch equations**

**Linear resistor**

Let us consider a linear resistor as shown in Fig. 15.6.

![Fig. 15.6. A linear resistor](image)

The resistor is described by Ohm’s law

\[ v(t) = R i(t) , \]  

(15.15)

where we assume that \( i(t) \) and \( v(t) \) are sinusoids at the same frequency \( \omega \):

\[ v(t) = V_m \cos(\omega t + \phi_v) , \quad i(t) = I_m \cos(\omega t + \phi_i) . \]

We form the corresponding phasors \( V = V_m e^{j\phi_v} \), \( I = I_m e^{j\phi_i} \) and formulate the equation

\[ V = RI . \]  

(15.16)

Below it is proved that equation (15.16) implies equation (15.15).

Let us multiply both sides of equation (15.16) by \( e^{j\omega t} \)

\[ V e^{j\omega t} = R I e^{j\omega t} \]  

(15.17)

and present the phasor \( V \) and \( I \) in the polar form

\[ V_m e^{j\phi_v} e^{j\omega t} = R I_m e^{j\phi_i} e^{j\omega t} . \]
After simple rearrangement we have
\[ V_m e^{j(\omega t + \phi_v)} = RI_m e^{j(\omega t + \phi_i)} \].

(15.18)

Next, we apply Euler’s formula finding
\[ V_m \cos(\omega t + \phi_v) + jV_m \sin(\omega t + \phi_v) = RI_m \cos(\omega t + \phi_i) + jRI_m \sin(\omega t + \phi_i) \].

(15.19)

Since equation (15.19) holds if the real parts on the left and right side and the imaginary parts on the left and right side are identical we may write
\[ V_m \cos(\omega t + \phi_v) = RI_m \cos(\omega t + \phi_i) \],

which means
\[ v(t) = Ri(t) \].

Thus, in the sinusoidal steady-state Ohm’s equation describing a linear resistor can be written down directly in terms of phasors, replacing any sinusoid by its phasor.

**Linear inductor**

Figure 15.7 shows a linear inductor. We assume that the voltage \( v(t) \) and the current \( i(t) \) are sinusoids at the same frequency \( \omega \):
\[ v(t) = V_m \cos(\omega t + \phi_v), \quad i(t) = I_m \cos(\omega t + \phi_i) \].

![Fig. 15.7. A linear inductor](image)

The corresponding phasors are as follows
\[ V = V_m e^{j\phi_v}, \quad I = I_m e^{j\phi_i} \].
In the time domain an inductor is described by the equation
\[ v(t) = L \frac{di(t)}{dt}. \]  
(15.20)

We write the corresponding equation in terms of phasors
\[ V = j \omega LI \]  
(15.21)

and prove that equation (15.21) implies equation (15.20). For this purpose we multiply both sides of equation (15.21) by $e^{j\omega t}$
\[ V e^{j\omega t} = j \omega L I e^{j\omega t}. \]  
(15.22)

Since $j = e^{j\frac{\pi}{2}}$ we rewrite equation (15.22) in the form
\[ V_m e^{j(\omega t + \phi_v)} = \omega LI_m e^{j\left(\omega t + \phi_v + \frac{\pi}{2}\right)}. \]  
(15.23)

Next, we apply Euler’s formula to the exponential functions
\[ V_m \cos(\omega t + \phi_v) + j V_m \sin(\omega t + \phi_v) = \]  
\[ = \omega LI_m \cos\left(\omega t + \phi_v + \frac{\pi}{2}\right) + j \omega LI_m \sin\left(\omega t + \phi_v + \frac{\pi}{2}\right). \]  
(15.24)

Equation (15.24) is satisfied if the real parts on the left and right side are identical. Hence, we may write
\[ V_m \cos(\omega t + \phi_v) = \omega LI_m \cos\left(\omega t + \phi_v + \frac{\pi}{2}\right) = \]  
\[ = -\omega LI_m \sin(\omega t + \phi_v) = L \frac{d}{dt} \left( I_m \cos(\omega t + \phi_v) \right), \]  
or
\[ v(t) = L \frac{di(t)}{dt}. \]  

Notes

114
Thus, equation (15.21) implies equation (15.20).

**Linear capacitor**

![Diagram of a linear capacitor](image)

In the time domain a linear capacitor is described by the equation

\[ i(t) = C \frac{dv(t)}{dt}. \]  

(15.25)

The corresponding equation in terms of phasors is

\[ I = j\omega CV. \]  

(15.26)

It can be rewritten in the form

\[ V = \frac{1}{j\omega C} I = -\frac{j}{\omega C} I. \]  

(15.27)

The crucial point is that, in terms of phasors, branch equations (15.15), (15.21) and (15.27) become algebraic equations with complex coefficients.

### 15.4 Impedance and admittance

Consider the circuit shown in Fig. 15.9, where the one-port consists of linear resistors, inductors and capacitors. Let the port voltage and the port current be sinusoidal functions of time. Their phasors are indicated in this figure.

We define the impedance of the one-port as the ratio of the voltage phasor \( V \) and the current phasor \( I \), that is

\[ Z = \frac{V}{I}. \]  

(15.28)
Fig. 15.9. A linear one-port

Setting $V = |V|e^{j\phi}$ and $I = |I|e^{j\phi}$ we have

$$Z = |Z|e^{j\phi}$$  \hspace{1cm} (15.29)

where

$$|Z| = \frac{|V|}{|I|} \quad \text{and} \quad \phi = \phi_v - \phi_i .$$

The impedance $Z$ given by the expression (15.28) can be transformed into rectangular representation using Euler’s formula. As a result we obtain

$$Z = |Z|\cos\phi + j|Z|\sin\phi .$$  \hspace{1cm} (15.30)

Denoting

$$Z = |Z|\cos\phi = R \quad \text{and} \quad Z = |Z|\sin\phi = X$$

we have

$$Z = R + jX ,$$  \hspace{1cm} (15.31)
where $R$ is called resistance or the resistive part of the impedance and $X$ is called reactance or the reactive part of the impedance. Both $R$ and $X$ are measured in ohms. On the basis of (15.31) we find the magnitude of the impedance

$$|Z| = \sqrt{R^2 + X^2}$$

and the phase angle

$$\phi = \tan^{-1} \frac{X}{R}.$$  \hspace{1cm} (15.33)

The relationships (15.32) and (15.33) enable us to create a triangle in the complex plane, shown in Fig. 15.10. It is called an impedance triangle.

Let us consider the series connection of two circuit elements specified by their impedances $Z_1$ and $Z_2$ (see Fig. 15.11).

![Impedance Triangle](image)

**Fig. 15.10. The impedance triangle**

Let us consider the series connection of two circuit elements specified by their impedances $Z_1$ and $Z_2$ (see Fig. 15.11).

![Series Connection](image)

**Fig. 15.11. Series connection of two elements**
We write KVL equation

\[ V = V_1 + V_2 \]

and find

\[ Z = \frac{V}{I} = \frac{V_1 + V_2}{I} = Z_1 + Z_2. \]

Thus, the impedance of two elements connected in series is the sum of their impedances.

The reciprocal of the impedance \( Z \), labeled \( Y \), is called the admittance

\[ Y = \frac{1}{Z}. \quad (15.34) \]

The units of admittance are siemens, abbreviated to S. Substituting (15.29) into (15.34) gives

\[ Y = \frac{1}{Z} = \frac{1}{|Z| e^{j\phi}} = \frac{1}{|Z|} e^{-j\phi}. \quad (15.35) \]

Thus, the magnitude of the admittance is equal to the inverse of the magnitude of the impedance

\[ |Y| = \frac{1}{|Z|}, \quad (15.36) \]

whereas the phase of the admittance equals the phase of the impedance with the opposite sign \((-\phi)\).

Plugging (15.31) into (15.34) yields

\[ Y = \frac{1}{R + jX} = \frac{R - jX}{R^2 + X^2} = \frac{R}{R^2 + X^2} - j \frac{X}{R^2 + X^2}. \]
This expression can be presented in the compact form

\[ Y = G + jB \]

where \( G = \frac{R}{R^2 + X^2} \) is called conductance and \( B = -\frac{X}{R^2 + X^2} \) is called susceptance. The units of \( G \) and \( B \) are siemens.

If the reactance is positive \((X > 0)\), the element is called inductive, if \( X < 0 \) the element is called capacitive. Since

\[ B = -\frac{X}{R^2 + X^2} \]

we conclude that the element is inductive when \( B \) is negative and capacitive when \( B \) is positive.

On the basis of the equation

\[ Y = G + jB \]

we obtain

\[ |Y| = \sqrt{G^2 + B^2} \quad \text{and} \quad \chi Y = -\phi = \tan^{-1} \left( \frac{B}{G} \right). \]

These relationships are summarized graphically in the complex plane as shown in Fig. 15.12. The construction is called an admittance triangle.

![Admittance Triangle](image)
Consider two elements, specified by their admittances $Y_1$ and $Y_2$, connected in parallel (see Fig. 15.13).

![Fig. 15.13. Two elements connected in parallel](image)

Using KCL we obtain

$$I = I_1 + I_2.$$ 

Hence, it holds

$$Y = \frac{I}{V} = \frac{I_1 + I_2}{V} = Y_1 + Y_2.$$ 

The example given below explains how to use the concept of impedance and admittance in the analysis of linear circuits in the sinusoidal steady-state. The analysis of sinusoidal circuits using phasors is called the analysis in the frequency domain.

**Example 15.3**

![Fig. 15.14. An example circuit](image)
Consider the circuit shown in Fig. 15.14, driven by the sinusoidal voltage source
\[ v_S(t) = 14.14 \cos(1000t + 45^\circ) \text{V} . \]

We wish to find the steady state current \( i_1(t) \) using the phasor concept.

The phasor of \( v_S(t) \) is
\[ V_S = 14.14 e^{j45^\circ} \text{V} . \]

Figure 15.15 shows the circuit in a phasor format.

![Fig. 15.15. The circuit of Fig. 15.14 in phasor format](image)

We compute the impedance faced by the voltage source. Since the elements specified by impedances \( Z_2 \) and \( Z_C \) are connected in parallel we find the admittance
\[ Y_{2C} = Y_2 + Y_C = \frac{1}{Z_2} + \frac{1}{Z_C} . \]

Hence, the impedance \( Z_{2C} \) is
\[ Z_{2C} = \frac{1}{Y_{2C}} = \frac{Z_2 Z_C}{Z_2 + Z_C} = \frac{R_2 \left( -\frac{j}{\omega C} \right)}{R_2 - \frac{j}{\omega C}} . \]
Next, we compute the total impedance

\[
Z = Z_1 + Z_{2C} = 40 \left( \frac{-j0.025}{0.025} \right) = 60 - j20 = 63.24e^{-j\theta} \Omega.
\]

Having the impedance we find

\[
I_1 = \frac{V_S}{Z} = \frac{14.14e^{j45^\circ}}{63.24e^{j18.43^\circ}} = 0.223e^{j63.43^\circ} A,
\]

hence, we have

\[
i_1(t) = 0.223\cos(1000t + 63.43^\circ) A.
\]

### 15.5 Phasor diagrams

Any phasor of a voltage and a current can be drawn as a vector having the length equal to the magnitude and the angle equal to the phase of the phasor. The diagram thus obtained is called a phasor diagram.

For the linear inductor shown in Fig. 15.16 we have

\[
I_L = \frac{1}{j\omega L} V_L = \frac{1}{\omega L} e^{-j\frac{\pi}{2}} V_L.
\]

![Fig. 15.16. A linear inductor](image)
Using the voltage phasor $V_L$ as the horizontal reference we draw the phasor $I_L$ vertically downward as shown in Fig. 15.17.

![Fig. 15.17. Phasor diagram of the inductor](image)

We say that the inductor current lags behind the inductor voltage by $\frac{\pi}{2}$.

For the capacitor shown in Fig 15.18 we write

$$V_C = -\frac{i}{\omega C}I_C = \frac{1}{\omega C}e^{-j\frac{\pi}{2}}I_C.$$  

![Fig. 15.18. A linear capacitor](image)

Assuming that $\phi_i = 0$ we draw the phasor diagram as depicted in Fig. 15.19.

![Fig. 15.19. Phasor diagram of the capacitor](image)
We say that capacitor voltages lag behind the capacitor current by $\frac{\pi}{2}$.

Now we draw a phasor diagram showing the current and voltages in the circuit shown in Fig. 15.20.

Using KVL we write

$$V_S = V_1 + V_2 + V_3.$$ 

We use a current phasor as the horizontal reference. Since $V_1 = j\alpha L_1 i = \alpha L_1 I e^{j\frac{\pi}{2}}$, the voltage $V_1$ leads the current by $\frac{\pi}{2}$. Consequently the phasor $V_1$ is drawn vertically upward (see Fig. 15.21).
The voltage \( V_2 = R_2 I \) has the same phase as the current, hence, the phasor \( V_2 \) is horizontal, similarly as \( I \). We shift this phasor to the tip of \( V_1 \) as depicted in Fig. 15.22.

![Fig. 15.22. A part of the phasor diagram of the circuit shown in Fig. 15.20](image)

Since \( V_3 = \frac{1}{j\omega C_3} I = -\frac{j}{\omega C_3} I = \frac{1}{\omega C_3} e^{-j\theta_2} \), the phasor \( V_3 \) is drawn vertically downward and shifted to the tip of \( V_2 \). Next, the three phasors are added vectorially to obtain the voltage \( V_S \) as illustrated in Fig. 15.23.

![Fig. 15.23. A phasor diagram of the circuit of Fig. 15.20](image)

### 15.6 Effective value

A commonly used measure of sinusoidal voltages and currents is the effective value also called a root mean square value (abbreviated to rms).
For a sinusoidal voltage

\[ v(t) = V_m \cos(\omega t + \phi_v) \]

it is defined as follows

\[ V_{\text{eff}} = V_{\text{rms}} = \sqrt{\frac{1}{T_0} \int_0^{T_0} (v(t))^2 \, dt} = \sqrt{\frac{1}{T_0} \int_0^{T_0} (V_m \cos(\omega t + \phi_v))^2 \, dt} \quad \text{. (15.37)} \]

To integrate, we apply the trigonometric identity

\[ \cos^2 \alpha = \frac{1}{2} (1 + \cos2\alpha) \]

finding

\[ \int_0^T V_m^2 \cos^2 (\omega t + \phi_v) \, dt = V_m^2 \int_0^T \frac{1}{2} (1 + \cos(2\omega t + 2\phi_v)) \, dt = \]

\[ = \frac{1}{2} V_m^2 T + \frac{1}{2} V_m^2 \int_0^T \cos(2\omega t + 2\phi_v) \, dt \quad \text{. (15.38)} \]

In the interval \(0 - T\) there are exactly two periods of the function \(\cos(2\omega t + 2\phi_v)\), consequently

\[ \int_0^T \cos(2\omega t + 2\phi_v) \, dt = 0 \quad \text{.} \]

Hence, we have

\[ \int_0^T V_m^2 \cos^2 (2\omega t + 2\phi_v) \, dt = \frac{1}{2} V_m^2 T \]

and
Similarly we define the effective (rms) value of the sinusoidal current

\[ i(t) = I_m \cos(\omega t + \phi_i) \]

finding

\[ I_{\text{eff}} = I_{\text{rms}} = \frac{I_m}{\sqrt{2}} \].

\[ (15.40) \]
16. Power in sinusoidal steady state

16.1 The instantaneous and average power

Power is a very important quantity in electrical engineering, electronics, communications, and power systems. Hence, the question arises how to define power in sinusoidal circuits.

Consider a one-port consisting of resistors, inductors and capacitors (see Fig. 16.1) with sinusoidal port voltage

\[ v(t) = V_m \cos(\omega t + \phi_v) \, \text{V} \]  

(16.1)

and port current

\[ i(t) = I_m \cos(\omega t + \phi_i) \, \text{A} \]  

(16.2)

The instantaneous power received by the one-port is given by

\[ p(t) = v(t)i(t) = V_m I_m \cos(\omega t + \phi_v)\cos(\omega t + \phi_i) \, \text{W} \]  

(16.3)
Using the trigonometric identity

\[ \cos x \cos y = \frac{1}{2} (\cos(x + y) + \cos(x - y)) \]

we obtain

\[ p(t) = \frac{V_m I_m}{2} \cos(2\omega t + \phi_v + \phi_i) + \frac{V_m I_m}{2} \cos(\phi_v - \phi_i) . \] (16.4)

Thus, the instantaneous power is the sum of two components. The first one is a sinusoid whose frequency is twice of the voltage and current. The second component is a constant. Thus, the waveform of \( p(t) \) is a sinusoid having the angular frequency \( 2\omega \) shifted upward by the value equal to the second term.

Average power, labeled \( P_{av} \), is defined as the mean value of the instantaneous power over an interval of \( T \), where \( T \) is a period of the voltage or current

\[ P_{av} = \frac{1}{T} \int_0^T p(t) dt . \] (16.5)

Substituting (16.4) into (16.5) yields

\[ P_{av} = \frac{1}{T} \int_0^T \left( \frac{V_m I_m}{2} \cos(2\omega t + \phi_v + \phi_i) + \frac{V_m I_m}{2} \cos(\phi_v - \phi_i) \right) dt . \]

The integral of the sinusoid \( \cos(2\omega t + \phi_v + \phi_i) \), having the period \( \frac{T}{2} \), over the interval \( 0 \) to \( T \) is zero, because the total area under the waveform of this sinusoid is equal to zero. Hence, we have

\[ P_{av} = \frac{1}{T} \int_0^T \frac{V_m I_m}{2} \cos(\phi_v - \phi_i) dt = \frac{V_m I_m}{2} \cos \phi = V_{eff} I_{eff} \cos \phi , \] (16.6)
where \( \phi = \phi_v - \phi_i \) is called a power factor. Note that \( \phi \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \), \( \cos \phi \geq 0 \) and the average power is a real, nonnegative number.

### 16.2 The complex power

Another power useful in the AC analysis of circuits is the complex power defined as follows

\[
P = \frac{1}{2} V^* I^* ,
\]

(16.7)

where \( V \) is the phasor of the port voltage whereas \( I^* \) is the complex conjugate of the port current phasor. Hence, it holds

\[
P = \frac{1}{2} |V| |I| e^{j(\phi_v - \phi_i)} = \frac{1}{2} |V||I| e^{j(\phi_v - \phi_i)} = \frac{1}{2} |V||I| \cos(\phi_v - \phi) + j \frac{1}{2} |V||I| \sin(\phi_v - \phi) = \frac{1}{2} V_m I_m \cos \phi + \frac{j}{2} V_m I_m \sin \phi = V_{\text{eff}} I_{\text{eff}} \cos \phi + j V_{\text{eff}} I_{\text{eff}} \sin \phi.
\]

Thus, the complex power is a complex number. The real part of this number is, according to (16.6), the average power

\[
P_{\text{av}} = \text{Re}(P).
\]

Let \( Z = R + jX \) and \( Y = G + jB \) be the impedance and admittance of the one-port, respectively (see Fig. 16.1). Then using (16.7) we obtain

\[
P_{\text{av}} = \text{Re}(P) = \text{Re} \left( \frac{1}{2} Z I I^* \right) = \text{Re} \left( \frac{1}{2} |I|^2 (R + jX) \right) = \frac{1}{2} |I|^2 R = \frac{1}{2} I_m^2 R = I_{\text{eff}}^2 R.
\]

(16.9)
The imaginary part of the complex power, labeled $P_x$, is called a reactive power, that is (see (16.8))

$$P_x = \text{Im}(P) = \frac{1}{2} V_m I_m \sin \phi = V_{\text{eff}} I_{\text{eff}} \sin \phi .$$  

Similarly, as in the case of the average power, we find some alternative expression for the reactive power

$$P_x = \text{Im}(P) = \text{Im} \left( \frac{1}{2} Z I I^* \right) = \text{Im} \left[ \frac{1}{2} (R + jX) |I|^2 \right] =$$

$$= \text{Im} \left[ \frac{1}{2} R |I|^2 + jX |I|^2 \right] = \frac{1}{2} X |I|^2 = \frac{1}{2} X I_m^2 = I_{\text{eff}}^2 X .$$

or

$$P_x = \text{Im}(P) = \text{Im} \left[ \frac{1}{2} V (Y V)^* \right] = \text{Im} \left[ \frac{1}{2} Y^* V V^* \right] =$$

$$= \text{Im} \left[ \frac{1}{2} (G - jB) |V|^2 \right] = -\frac{1}{2} B |V|^2 = -\frac{1}{2} B V_m^2 = -BV_{\text{eff}}^2 .$$

Thus, the complex power is the complex number whose real part is the average power and the imaginary part is the reactive power

$$P = P_{\text{av}} + jP_x .$$  

Using (16.8) we obtain
\[ P = \frac{1}{2} |V| |I| \cos \phi + j \frac{1}{2} |V| |I| \sin \phi = \frac{1}{2} |V| |I| (\cos \phi + j \sin \phi). \]

By virtue of Euler’s formula it holds
\[ P = \frac{1}{2} |V||I| e^{j\phi} = \frac{1}{2} V_m I_m e^{j\phi} = V_{\text{eff}} I_{\text{eff}} e^{j\phi}. \]  \hspace{1cm} (16.15)

Expression (16.14) and (16.15) lead to the equations
\[ |P|^2 = P_{\text{av}}^2 + P_x^2 \]  \hspace{1cm} (16.16)

and
\[ \tan \phi = \frac{P_x}{P_{\text{av}}}. \]  \hspace{1cm} (16.17)

They enable us to create the geometrical construction called a power triangle. Two variants of the power triangle are shown in Fig. 16.2.

Consider the one-port consisting of resistors, inductors and capacitors, supplied with a sinusoidal source \( v(t) \) as shown in Fig. 16.3.
Fig. 16.3. One-port RLC driven by a sinusoidal voltage source

It can be proved that the complex power $P$ of the one-port is equal to the sum of the complex powers $P_k$ of all elements of the one-port, i.e.

$$P = \sum_{k=1}^{b} P_k,$$  \hspace{1cm} (16.18)

where $b$ is the number of the elements.

Since

$$P = P_{av} + jP_x$$  and  $$P_k = (P_{av})_k + j(P_x)_k, \quad k = 1, \ldots, b,$$

equation (16.18) leads to two equations

$$P_{av} = \sum_{k=1}^{b} (P_{av})_k,$$  \hspace{1cm} (16.19)

$$P_x = \sum_{k=1}^{b} (P_x)_k,$$  \hspace{1cm} (16.20)
16.3 Measurement of the average power

To measure the average power we use a wattmeter. The wattmeter is an instrument having a current coil and a voltage coil. To measure the average power of a load the coils are connected as shown in Fig. 16.4.

\[
\frac{1}{2} \text{Re}(VI^*) = V_{\text{eff}} I_{\text{eff}} \cos \angle (I, V), \quad (16.21)
\]

where \( I \) and \( V \) are phasor of the sinusoidal voltage \( v(t) \) and current \( i(t) \), respectively. For an ideal wattmeter the impedance of the current coil tends to zero and the impedance of the voltage coil tends to infinity. As a result, the ideal wattmeter does not introduce any disturbance in the circuit. Both the current and the voltage remains the same as without the wattmeter, \( \cos \angle (I, V) = \phi \) and the wattmeter reading is the average power consumed by the load.

![Fig. 16.4. Measurement of the average power](image-url)
16.4 Theorem on the maximum power transfer

Consider the circuit consisting of a sinusoidal voltage source, represented by phasor $V_S$ and impedance $Z_S$, and the load represented by impedance $Z_L$ (see Fig. 16.5). We want to find $Z_L$ so that the average power entering the load is maximum.

![Fig. 16.5. Simple circuit terminated by load $Z_L$](image)

Let the impedances $Z_S$ and $Z_L$ be given in rectangular form

$$Z_S = R_S + jX_S, \quad Z_L = R_L + jX_L.$$

On the basis of (16.9) the average power delivered to the load is

$$P_{av} = \frac{1}{2} |I|^2 R_L.$$

Since

$$I = \frac{V_S}{Z_S + Z_L},$$

it follows

$$P_{av} = \frac{1}{2} \frac{V_S}{|Z_S + Z_L|^2} R_L = \frac{1}{2} \frac{|V_S|^2}{|Z_S + Z_L|^2} R_L = \frac{1}{2} \frac{|V_S|^2}{|R_S + jX_S + R_L + jX_L|^2} R_L = \frac{1}{2} \frac{|V_S|^2 R_L}{(R_S + R_L)^2 + (X_S + X_L)^2}.$$
In the above expression we need to choose $R_L$ and $X_L$ to maximize $P_{av}$. Since $X_L$ can either be positive or negative we choose

$$X_L = -X_S.$$  

(16.22)

Consequently, the formula for $P_{av}$ becomes

$$P_{av} = \frac{1}{2} \left| V_S \right|^2 \frac{R_L}{(R_L + R_S)^2}.$$  

(16.23)

To determine $R_L$ corresponding to the optimum of $P_{av}$ we compute the derivative with respect to $R_L$

$$\frac{dP_{av}}{dR_L} = \frac{1}{2} \left| V_S \right|^2 \frac{(R_L + R_S)^2 - R_L 2(R_L + R_S)}{(R_L + R_S)^4}$$

and set it to zero

$$\frac{1}{2} \left| V_S \right|^2 \frac{R_L^2 + R_S^2 + 2R_LR_S - 2R_L^2 - 2R_LR_S}{(R_L + R_S)^4} = 0.$$  

The solution of this equation is

$$R_L = R_S.$$ 

It can be proved that the optimum of $P_{av}$ means the maximum. Thus, the maximum average power is attained when

$$Z_L = R_S - jX_S = Z_S^*$$  

(16.24)

and it is

$$\max P_{av} = \frac{1}{2} \left| V_S \right|^2 \frac{R_S}{(2R_S)^2} = \frac{\left| V_S \right|^2}{8R_S}.$$  

(16.25)
17. Resonant circuits

17.1 Series resonant circuit

Let us consider the circuit shown in Fig. 17.1 consisting of an inductor, a capacitor and a resistor connected in series, driven by a sinusoidal voltage source.

Suppose that the voltage source is

\[ v_s(t) = V_{s_m} \cos(\omega t + \phi_s), \]

then the phasor associated to this voltage equals

\[ V_s = V_{s_m} e^{j\phi_s}. \]

In Fig. 17.1 \( V_L, V_C, \) and \( V_R \) are the phasors of the voltages across the elements whereas \( I \) is the phasor of the current traversing all the elements. The circuit can be considered a one-port supplied with a voltage source. The impedance of this circuit is given by

\[
Z = R + j \omega L + j \left( \frac{1}{\omega C} \right) = R + j \left( \omega L - \frac{1}{\omega C} \right) = R + j X,
\]

(17.1)
where

\[ X = X(\omega) = \omega L - \frac{1}{\omega C} . \]  \hspace{1cm} (17.2)

The reactance \( X \) of the one-port is a monotonically increasing function of \( \omega \), for \( \omega \in (0, \infty) \). It becomes zero at \( \omega = \omega_0 \), i.e.

\[ \omega_0 L - \frac{1}{\omega_0 C} = 0 . \]  \hspace{1cm} (17.3)

Solving equation (17.3) for \( \omega_0 \) yields

\[ \omega_0 = \frac{1}{\sqrt{LC}} . \]  \hspace{1cm} (17.4)

The frequency \( \omega_0 \) is called a resonant frequency and the circuit at this frequency is said to be in resonance. Since \( X(\omega_0) = 0 \), the impedance in resonance is equal to the resistance

\[ Z_0 = R + jX(\omega_0) = R . \]  \hspace{1cm} (17.5)

Hence, the current in resonance is given by

\[ I_0 = \frac{V_s}{Z_0} = \frac{V_s}{R} . \]  \hspace{1cm} (17.6)

At an arbitrary frequency \( \omega \) the impedance of the one-port equals

\[ Z(\omega) = R + jX(\omega) . \]  \hspace{1cm} (17.7)

For each \( \omega \) the impedance is represented by a vector in the complex plain as shown in Fig. 17.2. As \( \omega \) varies in the interval \( (0, \infty) \) the tip of this vector traces a straight line parallel to the imaginary axis. At \( \omega = \omega_0 \) the reactance of this impedance equals zero and its magnitude is the smallest.
Thus, all the voltage from the source is across the resistor. At low frequencies $\omega L \ll \frac{1}{\omega C}$ and most of the voltage is across the capacitor. At high frequencies $\omega L \gg \frac{1}{\omega C}$ and most of the voltage is across the inductor.

![Diagram of Z(\omega) in the complex plane](image1)

Fig. 17.2. Plot of $Z(\omega)$ in the complex plane

Since at resonance $\omega L = \frac{1}{\omega C}$, the voltage across the inductor and the capacitor are as follows

$$V_L = j \omega_0 LI, \quad V_C = -j \frac{1}{\omega_0 C} I,$$

hence, $V_L = -V_C$. Thus, both these voltages have the same magnitude. It is illustrated in the phasor diagram shown in Fig. 17.3.

![Phasor diagram in resonance](image2)

Fig. 17.3. Phasor diagram in resonance
At resonance the ratio of the magnitude of the voltage across the inductor (or capacitor) to the magnitude of the source voltage is called a quality factor and labeled $Q$

\[ Q = \frac{V_m}{V_{s_m}} = \frac{V_{c_m}}{V_{s_m}} \quad \text{at} \quad \omega = \omega_0. \quad (17.8) \]

Since

\[ V_L = |V_L| = \omega_0 LI_m = V_C = \frac{1}{\omega_0 C} I_m \]

and

\[ V_S = |V_S| = R I_m, \]

then

\[ Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR}. \quad (17.9) \]

Now we consider the ratio of the phasor of the voltage across the resistor to the phasor of the source voltage at any frequency $\omega$. Since

\[ V_R = R I = R \frac{V_S}{R + j \left( \omega L - \frac{1}{\omega C} \right)}, \quad (17.10) \]

then it holds

\[ \frac{V_R}{V_S} = \frac{R}{R + j \left( \omega L - \frac{1}{\omega C} \right)}. \quad (17.11) \]

Expression (17.11) can be rearranged as follows

\[ \frac{V_R}{V_S} = \frac{1}{1 + j \left( \frac{L}{R} \left( \frac{\omega^2}{\omega_0^2} - \frac{\omega}{\omega_0} \right) \right)} = \frac{1}{1 + j \frac{\omega_0 L}{R} \left( \frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right)}. \quad (17.12) \]
Using (17.9) we have

\[ \frac{V_R}{V_S} = \frac{1}{1 + jQ\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)}, \]  

(17.13)

hence, it holds

\[ \frac{|V_R|}{|V_S|} = \frac{(V_m)_R}{(V_m)_S} = \frac{1}{\sqrt{1 + Q^2\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2}}. \]  

(17.14)

Plots of \( |V_R|/V_S \) versus \( \omega \), at different quality factors, are shown in Fig. 17.4.

Fig. 17.4. Plots of \( |V_R|/V_S \) against frequency \( \omega \)
At the resonance frequency $|V_R|/V_S$ is equal to 1, whereas at $\omega = 0$ and $\omega = \infty$ the ratio equals zero. Therefore the voltage across the resistor $R$ is the same as the source voltage at the resonant frequency. This voltage is slightly reduced for the frequencies close to the resonance frequency and considerably reduced at low and high frequencies. Such a circuit is called a bandpass filter.

The passband is defined as a range of the frequencies in the neighbourhood of the resonant frequency such that at the edges of the passband

$$\left(\frac{V_m}{V_m}\right)_R = \frac{1}{\sqrt{2}}.$$  \hspace{1cm} (17.15)

Substituting (17.14) into (17.15) gives

$$\frac{1}{\sqrt{1 + Q^2\left(\frac{\omega - \omega_0}{\omega}\right)^2}} = \frac{1}{\sqrt{2}}.$$ \hspace{1cm} (17.16)

We rearrange equations (17.16) as follows

$$1 + Q^2\left(\frac{\omega - \omega_0}{\omega_0}\right)^2 = 2, \quad Q^2\left(\frac{\omega - \omega_0}{\omega_0}\right)^2 = 1, \quad \frac{\omega - \omega_0}{\omega_0} = \pm \frac{1}{Q}.$$  

Hence, we have

$$\frac{\omega_2 - \omega_0}{\omega_0} = \frac{1}{Q}, \quad \frac{\omega_1 - \omega_0}{\omega_0} = \frac{1}{Q}.$$  

or

$$\omega_2^2 - \omega_0^2 = \frac{\omega_0 \omega_2}{Q}, \quad \omega_1^2 - \omega_0^2 = -\frac{\omega_0 \omega_1}{Q}.$$  

Notes
Next we subtract both sides of the above equations

$$\omega_2^2 - \omega_1^2 = \frac{\omega_0}{Q} (\omega_2 + \omega_1).$$

(17.17)

Dividing equation (17.17) by \((\omega_2 + \omega_1)\) we obtain the following expression for the passband

$$\omega_2 - \omega_1 = \frac{\omega_0}{Q}.$$  

(17.18)

The passband is illustrated in Fig. 17.5.
17.2 Parallel resonant circuit

Another type of a resonant circuit is the parallel resonant circuit shown in Fig. 17.6. The circuit is driven by sinusoidal source current represented by the phasor $I_S$. In Fig. 17.6 $I_C$, $I_L$, $I_R$ are phasors of the currents flowing through the circuit elements and $V$ is the phasor of the voltage across the parallel connection of these elements.

![Fig. 17.6. Parallel resonant circuit](image)

The admittance of the one-port faced by the source is

$$Y = G + j\omega C + \frac{1}{j\omega L} = G + j\left(\omega C - \frac{1}{\omega L}\right).$$  \hspace{1cm} (17.19)

The imaginary part of $Y$ (susceptance) is a function of the frequency $\omega$

$$B(\omega) = \omega C - \frac{1}{\omega L}. \hspace{1cm} (17.20)$$

The frequency $\omega = \omega_0$ such that $B(\omega_0) = 0$ is called a resonant frequency. To find this frequency, we solve the equation

$$\omega_0 C - \frac{1}{\omega_0 L} = 0,$$  \hspace{1cm} (17.21)

finding

$$\omega_0 = \frac{1}{\sqrt{LC}}.$$

(17.22)
At the resonance it holds

\[ I_S = YV = GV = I_R . \]  \hspace{1cm} (17.23)

Furthermore, since

\[ I_L = \frac{V}{j\omega_0 L} = -j \frac{1}{\omega_0 L} V \]  \hspace{1cm} (17.24)

and

\[ I_C = j\omega_0 CV , \]  \hspace{1cm} (17.25)

then

\[ I_L = -I_C \quad \text{or} \quad I_L + I_C = 0 . \]  \hspace{1cm} (17.26)

Thus, all the current from the current source goes through the resistor and the currents in the inductor and capacitor add to zero. The phasor diagram is shown in Fig. 17.7.

![Fig. 17.7. The phasor diagram of the resonant circuit](image1)

![Fig. 17.8. Parallel resonant circuit at \( \omega = \omega_0 \)](image2)

Figure 17.8 illustrates equation (17.26).

Let us consider the ratio of the currents \( I_R \) and \( I_S \) at an arbitrary frequency \( \omega \) and rearrange it as follows.
\[
\frac{I_R}{I_S} = \frac{GV}{YV} = \frac{G}{G + j\left(\frac{1}{\omega C} - \frac{1}{\omega L}\right)} = \frac{1}{1 + j\omega RC}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right).
\]

(17.27)

Since \(\frac{1}{G} = R\) and \(\frac{1}{LC} = \omega_0^2\), then we may write

\[
\omega_0 RC = \frac{\omega_0 C}{R} = \frac{\omega_0 C}{G} = \frac{\omega_0 C}{G\sqrt{\omega_0}} = \frac{(I_m)_C}{(I_m)_S}.
\]

(17.28)

The ratio of the current \((I_C)_m\) and \((I_S)_m\) is called a quality factor of the parallel resonant circuit and labeled \(Q\). Since at the resonance \((I_C)_m = (I_L)_m\), we have

\[
Q = \frac{(I_C)_m}{(I_S)_m} = \frac{(I_L)_m}{(I_S)_m} = \omega_0 CR = \frac{R}{\omega_0 L}.
\]

(17.29)

Substituting (17.29) into (17.27) yields

\[
\frac{I_R}{I_S} = \frac{1}{1 + jQ\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)}.
\]

(17.30)

Hence, we obtain the equation

\[
\left|\frac{I_R}{I_S}\right| = \left|\frac{(I_R)_m}{(I_S)_m}\right| = \frac{1}{\sqrt{1 + Q^2\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2}},
\]

(17.31)

illustrated in Fig. 17.8 for two values of \(Q\).
Fig. 17.9. Plots of \( \frac{|I_R|}{|I_S|} \) for different values of \( Q \)

The plots shown in Fig. 17.9 have the same shapes as in Fig. 17.5 and can be similarly interpreted. Thus, the parallel resonant circuit is a bandpass filter which passes the current through the resistor at the resonant frequency, slightly reduces the current in the neighbourhood of this frequency and considerably reduces it at a low frequency and at a high frequency.

Similarly, as in the case of a series resonant circuit, the passband is defined as a range of the frequencies in the neighbourhood of the resonant frequency such that at the edges of the passband it holds

\[
\frac{|I_R|}{|I_S|} = \frac{1}{\sqrt{2}}.
\]  

(17.32)

Hence, we obtain the following expression for the passband

\[
\omega_2 - \omega_1 = \frac{\omega_0}{Q}.
\]  

(17.33)
18. Coupled inductors

In this section we study the circuits in which two inductors are coupled magnetically.

18.1 Basic properties

Let us consider two coils, placed near each other, as shown in Fig. 18.1. If we supply the first coil with a generator so that the current $i_1$ varies with time and have the second coil open, then a time-varying magnetic flux $\phi$ is produced. It induces a voltage in the second coil.

![Fig. 18.1. Two magnetically coupled inductors](image)

Now we consider the case where currents $i_1$ and $i_2$ flow through the inductors. If the core is made of nonmetallic material, the following equations relate the currents and the fluxes leakages associated with the inductors.
\[
\begin{align*}
\phi_1 &= L_1 i_1 + M i_2, \\
\phi_2 &= M i_1 + L_2 i_2,
\end{align*}
\]

(18.1)

where \( L_1 \) is the self-inductance of the inductor 1, \( L_2 \) is the self-inductance of the inductor 2, and \( M \) is called a mutual inductance of the inductors 1 and 2. The equality of both mutual inductances is established in physics.

By Faraday’s law we have

\[
\begin{align*}
\nu_1 &= \frac{d\phi_1}{dt} = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}, \\
\nu_2 &= \frac{d\phi_2}{dt} = M \frac{di_1}{dt} + L_2 \frac{di_2}{dt}.
\end{align*}
\]

(18.2)

The mutual inductance \( M \) enables us to define the coefficient of coupling \( k \) as follows

\[
k = \frac{|M|}{\sqrt{L_1 L_2}}.
\]

(18.3)

It can be shown that \( k \leq 1 \).

Fig. 18.2. Coupled inductors with dotted terminals
The mutual inductance $M$ may be positive or negative. The sign of $M$ is specified by the reference directions of the currents and the physical arrangement of the coils. To determine the sign of $M$, we place a dot mark in each inductor so that, when both currents of the inductors enter them through the dotted terminals, then the fluxes due to these two currents add. This is illustrated in Fig. 18.2.

Next, we determine the sign of the mutual inductance $M$ as follows. We arbitrarily assign the reference directions of the currents $i_1$ and $i_2$. If the reference directions of these currents are such that both currents enter the inductors through the dotted terminals or leave the inductors through the dotted terminals, then $M$ is positive. If one current enters the inductor through the dotted terminal and the other leaves the inductor through the dotted terminal, then $M$ is negative.

Thus, if $M$ is positive, then both fluxes due to the currents $i_1$ and $i_2$ add. If $M$ is negative, they subtract.

According to the rule $M$ in Fig. 18.3a is positive, whereas in Fig. 18.3b is negative.

Fig. 18.3. Coupled inductors with positive (Fig. 18.3a) and negative (Fig. 18.3b) mutual inductance $M$

Dotted terminals enable us to specify the sign of $M$ without knowing the physical arrangement of the coils. Therefore, they are introduced in circuit diagrams as shown in Fig. 18.4.
Two coupled inductors are described in the time domain by the equations (18.2). If we consider sinusoidal steady state we formulate the corresponding equations in the phasor form

\[ V_1 = j\omega L_1 i_1 + j\omega M i_2, \quad (18.4) \]
\[ V_2 = j\omega M i_1 + j\omega L_2 i_2. \quad (18.5) \]

### 18.2 Connections of coupled inductors

#### Series connection

Figure 18.5 shows two coupled inductors connected in series. We consider the sinusoidal steady state and apply the phasor concept.
The voltage $V$ across the connection is given by

$$V = V_1 + V_2 = (R_1I + j\omega L_1I + j\omega MI) + (R_2I + j\omega L_2I + j\omega MI) =$$

$$= [R_1 + R_2 + j\omega(L_1 + L_2 + 2M)]I.$$

Hence, we obtain the impedance of this connection, as follows

$$Z = \frac{V}{I} = R_1 + R_2 + j\omega(L_1 + L_2 + 2M) = R_{eq} + j\omega L_{eq}, \quad (18.6)$$

where

$$R_{eq} = R_1 + R_2 \quad (18.7)$$

and

$$L_{eq} = L_1 + L_2 + 2M. \quad (18.8)$$

Equation (18.6) states that the series connection shown in Fig. 18.5 can be replaced by the equivalent circuit shown in Fig. 18.6.

![Equivalent Circuit](image)

Fig. 18.6. Equivalent circuit to the circuit depicted in Fig. 18.5

Note that the equivalent inductance is given by equation (18.8), where $M$ can be either positive or negative.

Figure 18.7a shows a phasor diagram for $M > 0$, whereas Fig. 18.8b shows a phasor diagram for $M < 0$. 
Fig. 18.7. Phasor diagrams of the connections shown in Fig. 18.5

**Parallel connection**
Figure 18.8 shows two coupled inductors connected in parallel.
We analyse this connection in the sinusoidal steady state using the phasor concept. As a result, we write the equations

\[ V = j\omega L_1 I_1 + j\omega M I_2, \]  
\[ V = j\omega L_2 I_2 + j\omega M I_1. \]  

(18.9)
\( (18.10) \)

Next, we solve equation (18.9) for \( I_2 \)

\[ I_2 = \frac{V}{j\omega M} - \frac{L_1}{M} I_1 \]  

(18.11)

and substitute (18.11) into (18.10)

\[ V = j\omega L_2 \frac{V}{j\omega M} - j\omega L_2 \frac{L_1}{M} I_1 + j\omega M I_1. \]

Hence, we find the current \( I_1 \)

\[ I_1 = \frac{1 - \frac{L_2}{M}}{j\omega \left( M - \frac{L_1 L_2}{M} \right)} V = \frac{M - L_2}{j\omega \left( M^2 - L_1 L_2 \right)} V. \]  

(18.12)

Similarly, we obtain

\[ I_2 = \frac{M - L_1}{j\omega \left( M^2 - L_1 L_2 \right)} V. \]  

(18.13)

Taking into account (18.12) and (18.13) we have

\[ I = I_1 + I_2 = \frac{-L_1 - L_2 + 2M}{j\omega \left( M^2 - L_1 L_2 \right)} V. \]  

(18.14)
Equation (18.14) leads to
\[ V = j \omega L_{eq} I, \quad (18.15) \]
where
\[ L_{eq} = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}. \quad (18.16) \]

Thus, the parallel connection of the coupled inductors, depicted in Fig. 18.8, is equivalent to the single inductor having the inductance specified by (18.16).

To illustrate the method for the analysis of circuits containing coupled inductors, we consider the circuit shown in Fig. 18.9.

First we write the equations for \( V_1 \) and \( V_2 \)
\[ V_1 = j \omega L_1 I_1 + j \omega M I_2, \quad (18.17) \]
\[ V_2 = j \omega L_2 I_2 + j \omega M I_1. \quad (18.18) \]

Then, we apply KVL in the two loops
\[ V_S = R_1 I_1 + V_1, \quad (18.19) \]
\[ R_L I_2 + V_2 = 0. \quad (18.20) \]
Using equations (18.17) – (18.20) we write

\[ V_S = (R_1 + j\omega L_1)I_1 + j\omega M I_2 , \quad (18.21) \]

\[ (R_L + j\omega L_2)I_2 + j\omega M I_1 = 0 . \quad (18.22) \]

Next, we rearrange the set of equations (18.21) – (18.22) as follows

\[ I_2 = -\frac{j\omega M}{R_L + j\omega L_2} I_1 , \quad (18.23) \]

\[ V_S = \left( R_1 + j\omega L_1 + \frac{\omega^2 M^2}{R_2 + j\omega L_2} \right) I_1 . \quad (18.24) \]

Hence, the impedance \( Z = V_S / I_1 \) is given by

\[ Z = R_1 + j\omega L_1 + \frac{\omega^2 M^2}{R_2 + j\omega L_2} . \quad (18.25) \]

### 18.3 Ideal transformer

Let us consider two coils wound around a ferromagnetic core. In such a case, both coils are linked by essentially the same magnetic flux. The time varying flux induces voltages across the coils which are approximately proportional to the number of turns.

![Fig. 18.10. Symbol of the ideal transformer](image-url)
In the ideal transformer, whose symbol is shown in Fig. 18.10, it holds

\[
\frac{v_2}{v_1} = \frac{N_2}{N_1},
\]

(18.26)

where \( N_1 \) and \( N_2 \) are the numbers of the coils turns.

Another equation describing the ideal transformer refers to the current ratio

\[
\frac{i_2}{i_1} = \frac{N_1}{N_2}.
\]

(18.27)

Equation (18.26) states that depending on \( N_2/N_1 \) the voltage in the ideal transformer is stepped up or stepped down. Using equations (18.26) and (18.27) we see that a step up in voltage accompanies a step down in current. Consequently, there is no power loss in the ideal transformer.

Consider the ideal transformer, in the sinusoidal steady state, terminated by impedance \( Z_0 \), as shown in Fig. 18.11.

![Figure 18.11](image.png)

In such a case we use the phasor concept and rewrite equations (18.26) and (18.27) in the form
\[
\frac{V_2}{V_1} = \frac{N_2}{N_1}, \quad (18.28)
\]
\[
\frac{I_2}{I_1} = -\frac{N_1}{N_2}. \quad (18.29)
\]

On the basis of (18.28) and (18.29) we obtain
\[
\frac{V_1}{I_1} = \left(\frac{N_1}{N_2}\right)^2 \frac{V_2}{I_2}. \quad (18.30)
\]

Since
\[
Z_0 = \frac{V_2}{(-I_2)},
\]
then
\[
Z_1 = \frac{V_1}{I_1} = \left(\frac{N_1}{N_2}\right)^2 Z_0. \quad (18.31)
\]

Equation (18.31) expresses the impedance at the primary \( Z_1 \) in terms of impedance \( Z_0 \).
19. Three-phase systems

19.1 Introduction

Three-phase systems are commonly used in the generation, transmission and distribution of electric power. Power in a three-phase system is constant rather than pulsating and three-phase motors start and run much better than single-phase motors. A three-phase system is a generator-load pair in which the generator produces three sinusoidal voltages of equal amplitude and frequency but differing in phase by 120° from each other.

The phase voltages \( v_a(t) \), \( v_b(t) \) and \( v_c(t) \) are as follows

\[
\begin{align*}
    v_a &= V_m \cos \omega t \\
    v_b &= V_m \cos \left( \omega t - 120^\circ \right) \\
    v_c &= V_m \cos \left( \omega t - 240^\circ \right)
\end{align*}
\]  

(19.1)

whereas the corresponding phasors are

\[
\begin{align*}
    V_a &= V_m \\
    V_b &= V_m e^{-j120^\circ} \\
    V_c &= V_m e^{-j240^\circ}
\end{align*}
\]  

(19.2)

A three-phase system is shown in Fig 19.1. In a special case, all impedances are identical

\[
Z_a = Z_b = Z_c = Z
\]  

(19.3)

Such a load is called a balanced load and is described by equations

\[
\begin{align*}
    I_a &= \frac{V_a}{Z} \\
    I_b &= \frac{V_b}{Z} \\
    I_c &= \frac{V_c}{Z}
\end{align*}
\]
Using KCL, we have
\[ I_n = I_a + I_b + I_c = \frac{1}{Z} (V_a + V_b + V_c) , \] (19.4)

where
\[ V_a + V_b + V_c = V_m \left( 1 + e^{-j120^\circ} + e^{-j240^\circ} \right) = \]
\[ = V_m \left( 1 + \cos 120^\circ - j \sin 120^\circ + \cos 240^\circ - j \sin 240^\circ \right) = V_m \left( 1 - \frac{1}{2} - j \frac{\sqrt{3}}{2} - \frac{1}{2} + j \frac{\sqrt{3}}{2} \right) = 0 . \]

Setting the above result into (19.4), we obtain
\[ I_n = 0 . \] (19.5)

Since the current flowing through the fourth wire is zero, the wire can be removed (see Fig. 19.2).

The system of connecting the voltage sources and the load branches, as depicted in Fig. 19.2, is called a Y-system or a star system. Point n is called the neutral point of the generator and point n' is called the neutral point of the load.
Each branch of the generator or load is called a phase. The wires connecting the supply to the load are called lines. In the Y-system shown in Fig. 19.2 each line current is equal to the corresponding phase current, whereas the line-to-line voltages (or simply line voltages) are not equal to the phase voltages.

19.2 **Y-connected systems**

We consider Y-connected generator sources (see Fig. 19.3).
The phasors of the phase voltages can be generally written as follows

\[ V_a = V = V_m e^{j\alpha} \]
\[ V_b = V e^{-j30^\circ} \quad \text{(19.6)} \]
\[ V_c = V e^{-j60^\circ} \]

We determine the line voltages \( V_{ab} \), \( V_{bc} \), \( V_{ca} \) (see Fig. 19.3). Using KVL, we obtain

\[ V_{ab} = V_a - V_b = V_a \left(1 + \frac{1}{2} + j\frac{\sqrt{3}}{2}\right) = V_a \left(\frac{3}{2} + j\frac{\sqrt{3}}{2}\right) = \]
\[ = V_a \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} e^{j\tan^{-1}\frac{\sqrt{3}}{3}} = V_a \sqrt{3} e^{j30^\circ}. \]

Thus,

\[ V_{ab} = V_a \sqrt{3} e^{j30^\circ}. \quad \text{(19.7)} \]

holds and similarly we obtain

\[ V_{bc} = V_b \sqrt{3} e^{j30^\circ} \quad \text{(19.8)} \]
\[ V_{ca} = V_c \sqrt{3} e^{j30^\circ}. \quad \text{(19.9)} \]

The phasor diagram showing the phase and line voltages is shown in Fig. 19.4. Thus, the line voltages \( V_{ab} \), \( V_{bc} \), \( V_{ca} \) form a symmetrical set of phasors leading by 30° the set representing the phase voltages and they are \( \sqrt{3} \) times greater

\[ |V_{ab}| = |V_{bc}| = |V_{ca}| = \sqrt{3}|V_a|. \quad \text{(19.10)} \]

The same conclusion is valid in the Y connected load (see Fig. 19.5).
Fig. 19.4. Phasor diagram

Fig. 19.5. Y-connected load
19.3 Three-phase systems calculations

When the three phases of the load are not identical, an unbalanced system is produced. An unbalanced Y-connected system is shown in Fig. 19.2. The system of Fig. 19.2 contains perfectly conducting wires connecting the source to the load. Now we consider a more realistic case where the wires are represented by impedances $Z_{p}$ and the neutral wire connecting $n$ and $n'$ is represented by impedance $Z_{n}$ (see Fig. 19.6).

![Diagram of a three-phase system with not perfectly conducting wires](image)

Fig. 19.6. A three-phase system with not perfectly conducting wires

Using the node $n$ as the datum, we express the currents $I_{a}$, $I_{b}$, $I_{c}$ and $I_{n}$ in terms of the node voltage $V_{n}$

\[ I_{a} = \frac{V_{a} - V_{n}}{Z_{a} + Z_{p}} , \tag{19.11} \]

\[ I_{b} = \frac{V_{b} - V_{n}}{Z_{b} + Z_{p}} , \tag{19.12} \]
Hence, we obtain the node equation

\[
\frac{V_n}{Z_n} = \frac{V_a - V_n}{Z_a + Z_p} - \frac{V_b - V_n}{Z_b + Z_p} - \frac{V_c - V_n}{Z_c + Z_p} = 0.
\]

Solving this equation for \( V_n \) we have

\[
V_n = \frac{V_a}{Z_a + Z_p} + \frac{V_b}{Z_b + Z_p} + \frac{V_c}{Z_c + Z_p}.
\]

The above relationships enable us to formulate a method for the analysis of three-phase systems. The method consists of three steps as follows:

(i) Determine \( V_n \) using (19.15).

(ii) Calculate the currents \( I_a, I_b, I_c \) and \( I_n \) applying (19.11) - (19.14).

(iii) Find the phase and line voltages using Kirchhoff’s and Ohm’s laws.

When the neutral wire is removed, the system contains three connecting wires and is called a three-wire system. In such a case, we set \( |Z_n| \to \infty \) into (19.15)

\[
V_n = \frac{V_a}{Z_a + Z_p} + \frac{V_b}{Z_b + Z_p} + \frac{V_c}{Z_c + Z_p}.
\]
The balanced system can be considered as a special case of the unbalanced system, where \( Z_a = Z_b = Z_c = Z \). Using (19.16), we obtain

\[
V_a = \frac{1}{Z + Z_p} \left( V_a + V_b + V_c \right) = \frac{3}{Z + Z_p} = 0.
\] (19.17)

Consequently, the relationships (19.11)-(19.13) reduce to

\[
I_a = \frac{V_a}{Z + Z_p},
\] (19.18)

\[
I_b = \frac{V_b}{Z + Z_p},
\] (19.19)

\[
I_c = \frac{V_c}{Z + Z_p}.
\] (19.20)

Since \( V_b = V_a e^{-j120^\circ} \) and \( V_c = V_a e^{-j240^\circ} \), we have \( I_b = I_a e^{-j120^\circ} \) and \( I_c = I_a e^{-j240^\circ} \). Hence, we need to calculate \( I_a \) only using (19.18), which can be executed applying the one-phase circuit, described by equation (19.18), shown in Fig. 19.7.

![Fig. 19.7. One-phase circuit](image)

This means that the analysis of a balanced three-phase system can be reduced to the analysis of one-phase system depicted in Fig. 19.7.
Example 19.1
Let us consider three-phase system shown in Fig. 19.8. The system is supplied with a balanced three-phase generator, whereas the load is unbalanced.

The effective value of the generator phase voltage is 220V, the impedance of any connecting wire is $Z_p = (2 + j2) \Omega$ and the phase impedances of the load are $Z_a = (2 + j4) \Omega$, $Z_b = (4 - j2) \Omega$, $Z_c = (2 + j4) \Omega$. We wish to determine the line currents.

Since the circuit shown in Fig. 19.8 is a three-wire system, we apply equation (19.16) to compute $V_n$. The generator phase voltages are

\[
V_a = 220\sqrt{2} \text{ V},
\]

\[
V_b = V_a e^{-j120^\circ} = 220\sqrt{2}\left(\frac{1}{2} - j\frac{\sqrt{3}}{2}\right) = (-155.56 - j269.44) \text{ V},
\]

\[
V_c = V_a e^{-j240^\circ} = 220\sqrt{2}\left(\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) = (-155.56 + j269.44) \text{ V}.
\]
Using (19.16), we find

\[
V_n = \frac{220 \sqrt{2}}{4 + j6} + \frac{(-155.56 - j269.44)}{6} + \frac{(-155.56 + j269.44)}{4 + j6} = (97.5 - j61.2) \text{V}.
\]

Next, we compute the line currents using (19.11) – (19.13)

\[
I_a = \frac{V_a - V_n}{Z_a + Z_p} = \frac{220 \sqrt{2} - 97.5 + j61.2}{2 + j4 + 2 + j2} = (23.49 - j19.94) \text{A},
\]

\[
I_b = \frac{V_b - V_n}{Z_b + Z_p} = \frac{-155.56 - j269.44 - 97.5 + j61.2}{4 - j2 + 2 + j2} = (-42.18 - j34.70) \text{A},
\]

\[
I_c = \frac{V_c - V_n}{Z_c + Z_p} = \frac{-155.56 + j269.44 - 97.5 + j61.2}{2 + j4 + 2 + j2} = (18.68 + j54.63) \text{A}.
\]

### 19.4 Power in three-phase circuits

In balanced systems, the average power consumed by each load branch is the same and given by

\[
\bar{P}_{av} = V_{eff} I_{eff} \cos \phi,
\]

where \(V_{eff}\) is the effective value of the phase voltage, \(I_{eff}\) is the effective value of the phase current and \(\phi\) is the angle of the impedance. The total average power consumed by the load is the sum of those consumed by each branch, hence, we have

\[
P_{av} = 3\bar{P}_{av} = 3V_{eff} I_{eff} \cos \phi.
\]

In balanced \(Y\) systems, the phase current has the same effective value as the line current \(I_{eff} = (I_{eff})_L\), whereas the line voltage has the effective value \((V_{eff})_L\) which is
\[ \sqrt{3} \text{ times greater than the effective value of the phase voltage, } (V_{\text{eff}})_L = \sqrt{3} V_{\text{eff}}. \] Hence, using (19.22), we obtain

\[ P_{av} = 3 \left( \frac{V_{\text{eff}}}{\sqrt{3}} \right) \left( I_{\text{eff}} \right)_L \cos \phi = \sqrt{3} \left( V_{\text{eff}} \right)_L \left( I_{\text{eff}} \right)_L \cos \phi. \] (19.23)

Similarly, we derive the reactive power

\[ P_x = \sqrt{3} \left( V_{\text{eff}} \right)_L \left( I_{\text{eff}} \right)_L \sin \phi. \] (19.24)

In unbalanced systems, we add the powers of each phase

\[ P_{av} = (V_{\text{eff}})_{a} (I_{\text{eff}})_{a} \cos \phi_{a} + (V_{\text{eff}})_{b} (I_{\text{eff}})_{b} \cos \phi_{b} + (V_{\text{eff}})_{c} (I_{\text{eff}})_{c} \cos \phi_{c} \] (19.25)

\[ P_x = (V_{\text{eff}})_{a} (I_{\text{eff}})_{a} \sin \phi_{a} + (V_{\text{eff}})_{b} (I_{\text{eff}})_{b} \sin \phi_{b} + (V_{\text{eff}})_{c} (I_{\text{eff}})_{c} \sin \phi_{c}. \] (19.26)

In order to measure the average power in a three-phase \( Y \)-connected load, we use three wattmeters connected as shown in Fig. 19.9.

The reading of the ideal wattmeter \( W_1 \) is

\[ P_{W_1} = \frac{1}{2} \text{Re}(V_a I_a^*) = \frac{1}{2} (V_m)_{a} (I_m)_{a} \cos \phi_{a} = (V_{\text{eff}})_{a} (I_{\text{eff}})_{a} \cos \phi_{a} = P_a. \]

Similarly, \( W_2 \) and \( W_3 \) measure the average power of the load branch \( b \) and \( c \), respectively. Thus, the sum of the three readings will give the total average power. This method of the average power measurement is valid for both balanced and unbalanced \( Y \)-connected loads. Note that in the case of a balanced \( Y \)-connected load all three readings are identical and therefore, we use only one wattmeter.

For measuring average power in a three-phase three-wire system, we can use a method applying two wattmeters. In this method two wattmeters are connected by choosing any one line as the common reference for the voltage coils of the wattmeters. The current coils are connected in series with the other two lines (see Fig. 19.10) and the asterisk terminals of each wattmeter are short-circuited (see Fig. 19.10).
Fig. 19.9. Three-wattmeter measurement system

Fig. 19.10. Two-wattmeter measurement system
The indications of the wattmeters are

\[ P_{W_1} = \frac{1}{2} \text{Re}(V_{ac} I_a^*) , \]  
\[ P_{W_2} = \frac{1}{2} \text{Re}(V_{bc} I_b^*) . \]  

(19.27)  
(19.28)

The Y-connected load is shown in Fig. 19.11.

Since \( V_{ac} = V_a - V_c \) and \( V_{bc} = V_b - V_c \), we obtain

\[ P_{W_1} = \frac{1}{2} \text{Re}((V_a - V_c) I_a^*) = \frac{1}{2} \text{Re}(V_a I_a^* - V_c I_a^*), \]  
\[ P_{W_2} = \frac{1}{2} \text{Re}((V_b - V_c) I_b^*) = \frac{1}{2} \text{Re}(V_b I_b^* - V_c I_b^*). \]

The sum of \( P_{W_1} \) and \( P_{W_2} \) gives

\[ P_{W_1} + P_{W_2} = \frac{1}{2} \text{Re}[V_a I_a^* + V_b I_b^* - V_c (I_a^* + I_b^*)]. \]  

(19.29)
The currents $I_a, I_b, I_c$ satisfy KCL

$$I_a + I_b + I_c = 0.$$  

Hence, it holds

$$I_a^* + I_b^* + I_c^* = 0,$$

or

$$I_a^* + I_b^* = -I_c^* = 0. \tag{19.30}$$

Substituting (19.30) into (19.29) we have

$$P_{W_1} + P_{W_2} = \frac{1}{2} \text{Re}[V_a I_a^* + V_b I_b^* + V_c I_c^*] = P_{av}. \tag{19.31}$$

Equation (19.31) says that the sum of the two wattmeters readings in a Y-connected system equals the total average power consumed by the load.

Let us consider a balanced Y-connected load and calculate the instantaneous power delivered by the generator to the load

$$p(t) = v_a(t)i_a(t) + v_b(t)i_b(t) + v_c(t)i_c(t), \tag{19.32}$$

where

$$v_a(t) = V_m \cos \omega t, \quad v_b(t) = V_m \cos (\omega t - 120^\circ), \quad v_c(t) = V_m \cos (\omega t - 240^\circ), \tag{19.33}$$

and

$$i_a(t) = I_m \cos (\omega t - \phi), \quad i_b(t) = I_m \cos (\omega t - 120^\circ - \phi), \quad i_c(t) = I_m \cos (\omega t - 240^\circ - \phi), \tag{19.34}$$

where $v_a(t)$, $v_b(t)$, $v_c(t)$ are the voltages of the load branches, $i_a(t)$, $i_b(t)$, $i_c(t)$ are the currents of the load branches and $\phi$ is the angle of the load impedance. We substitute (19.33)-(19.34) in (19.32)
\[ p(t) = V_m I_m [\cos \omega t \cos (\omega t - \phi) + \cos (\omega t - 120^\circ) \cos (\omega t - 120^\circ - \phi) + \cos (\omega t - 240^\circ) \cos (\omega t - 240^\circ - \phi)] \]

and use the trigonometric identity
\[ \cos x \cdot \cos y = \frac{1}{2} [\cos (x - y) + \cos (x + y)] , \]

finding
\[ p(t) = \frac{1}{2} V_m I_m \left[3\cos \phi + \cos (2\omega t - \phi) + \cos (2\omega t - 240^\circ - \phi) + \cos (2\omega t - 480^\circ - \phi)\right] . \]

Since
\[ \cos (2\omega t - \phi) + \cos (2\omega t - 240^\circ - \phi) + \cos (2\omega t - 480^\circ - \phi) = 0 , \]

we obtain
\[ p(t) = \frac{3}{2} V_m I_m \cos \phi = 3V_{\text{eff}} I_{\text{eff}} \cos \phi = P_{\text{av}} . \]  

(19.35)

Thus, the total instantaneous power \( p(t) \) delivered by a three-phase generator to the balanced load is constant and equals the average power consumed by the load.
Reference books