9. The Fourier Transform

9.1. Introduction

Up to this point we considered periodic signals. The signals can be expanded in Fourier series consisting of infinite number of harmonics. In this section we will study situations where the signals are not periodic. To extend the Fourier method we introduce the Fourier transform.

Let us consider a finite duration signal $x(t)$ defined for every t in the interval $[-\beta, \beta]$ and equal to zero outside this interval as depicted in Fig.9.1.

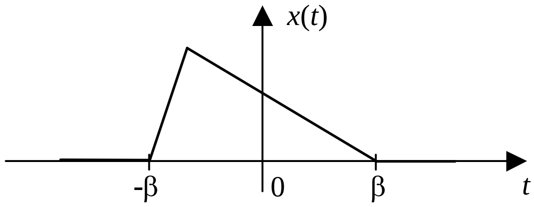

Fig. 9.1. A finite duration signal

Since the signal $x(t)$ is not periodic, it cannot be expanded into Fourier series. However, the Fourier series method can be used to represent $x(t)$ in any interval $[-\gamma, \gamma]$, where $\gamma \geq \beta$. For this purpose we create a periodic signal $\tilde{x}(t)$ with period 2γ which is identical to $x(t)$ for every t in $-\gamma < t < \gamma$ (see Fig.9.2).

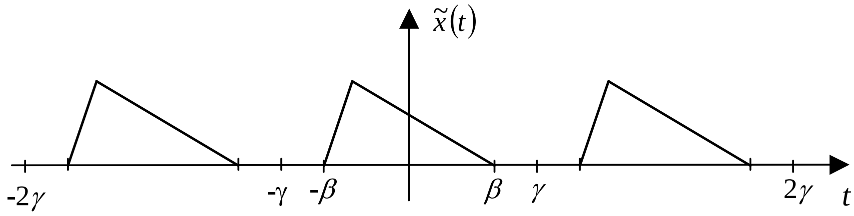

Fig. 9.2. A periodic signal that is equivalent to the signal of Figure 9.1 within the interval $[-\gamma, \gamma]$

Since $\tilde{x}(t)$ is periodic with period $T = 2\gamma$, it can be expanded in the exponential Fourier series. In the interval $[-\gamma, \gamma]$, the Fourier series expansion is $x(t)$. Thus,

$$x(t) = \sum_{n=-\infty}^{\infty} \tilde{c}_n \mathrm{e}^{\mathrm{j}n\omega_0 t} \qquad -\gamma < t < \gamma \qquad (9.1)$$

holds where

$$\omega_0 = \frac{2\pi}{T}$$

and the Fourier coefficients $\tilde{c}_n$ are uniquely determined by $x(t)$

$$\tilde{c}_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \mathrm{e}^{-\mathrm{j}n\omega_0 t} \mathrm{d}t \,. \qquad (9.2)$$

Let the integral (9.2) be denoted by $X(\mathrm{j}n\omega_0)$, i.e.

$$X(jn\omega_0) = \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \mathrm{e}^{-\mathrm{j}n\omega_0 t} \mathrm{d}t \,, \qquad (9.3)$$

then

$$\tilde{c}_n = \frac{1}{T} X(\mathrm{j}n\omega_0) = \frac{\omega_0}{2\pi} X(\mathrm{j}n\omega_0) \,. \qquad (9.4)$$

Inserting (9.4) into (9.1) yields

$$x(t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} X(jn\omega_0) \mathrm{e}^{\mathrm{j}n\omega_0 t} \omega_0 \qquad -\gamma < t < \gamma \,. \qquad (9.5)$$

It should be stressed that the two sides of equation (9.5) are equal for $-\gamma < t < \gamma$. Outside this interval $x(t)$ is zero and differs from the periodic signal specified by the right hand side.

Observe that as $\gamma \to \infty$ and consequently $T \to \infty$ the signal $\tilde{x}(t)$ is the same as $x(t)$. The right hand side of (9.5) gives the sum of exponential harmonics of frequencies $n\omega_0$, where $n = 0, \pm 1, \pm 2, \cdots$. As $T \to \infty$, $\omega_0 = \dfrac{2\pi}{T}$ tends to zero; hence, the distance between the frequencies of subsequent harmonics approaches zero. Thus, the discrete variable $n\omega_0$ becomes a continuous variable ω and ω_0 becomes dω. Consequently, the sum on the right hand side of (9.5) should be replaced by an integral. Thus, in the limit as $T \to \infty$, relationship (9.5) becomes

$$x(t) = \frac{1}{2\pi} \lim_{T\to\infty} \sum_{n=-\infty}^{\infty} X\left(\mathrm{j}n\omega_0\right)\mathrm{e}^{\mathrm{j}n\omega_0 t}\omega_0 = \\ = \frac{1}{2\pi}\int_{-\infty}^{\infty} X\left(\mathrm{j}\omega\right)\mathrm{e}^{\mathrm{j}\omega t}\mathrm{d}\omega . \tag{9.6}$$

To find $X(\mathrm{j}\omega)$, we use (9.3) repeated below

$$X\left(\mathrm{j}n\omega_0\right) = \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\mathrm{e}^{-\mathrm{j}n\omega_0 t}\mathrm{d}t$$

and assume that $T \to \infty$

$$X\left(\mathrm{j}\omega\right) = \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\mathrm{e}^{-\mathrm{j}n\omega_0 t}\mathrm{d}t = \int_{-\infty}^{\infty} x(t)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t . \tag{9.7}$$

Equations:

$$X\left(\mathrm{j}\omega\right) = \int_{-\infty}^{\infty} x(t)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t \tag{9.8}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X\left(\mathrm{j}\omega\right)\mathrm{e}^{\mathrm{j}\omega t}\mathrm{d}\omega \tag{9.9}$$

constitute the Fourier transform pair. $X(\mathrm{j}\omega)$ is called the Fourier transform of the time function $x(t)$, whereas $x(t)$ is the inverse Fourier transform of $X(\mathrm{j}\omega)$. The integral on the right hand side of (9.8) is called the Fourier integral.

Sufficient conditions for the existence of the Fourier transform are similar to the Dirichlet conditions for the Fourier series.

They are as follows:

(i) $x(t)$ must be absolutely integrable, i.e.

$$\int_{-\infty}^{\infty} |\, x(t)| \mathrm{d}t < \infty \,.$$

(ii) On any finite interval $x(t)$ has at most a finite number of maxima and minima.

(iii) On any finite interval $x(t)$ has at most a finite number of discontinuities and each of these discontinuities is finite.

The conditions are sufficient but not necessary. Consequently, many useful signals which do not meet them can also be analyzed using the Fourier transform.

Let us consider the condition (i). Unfortunately, many useful signals, e.g. the unit step function as well as periodic functions, are not absolutely integrable. It can be shown that any power signal (see Section 12) which meets the conditions (ii) and (iii) has a Fourier transform.

To determine the Fourier transform of a function, which is not absolutely integrable, we extend the idea of the Fourier transform as follows.

We multiply $x(t)$ by a factor $p(c,t)$ so that $p(0,t)=1$ and the integral

$$\int_{-\infty}^{\infty} |\, x(t)\, p(c,t)| \mathrm{d}t$$

is convergent. For instance, the factor can be chosen as

$$p(c,t) = \mathrm{e}^{-ct^2}\,, \qquad c > 0\,.$$

Next, we find the Fourier transform of $x(t)\,p(c,t)$. If the Fourier transform exists for any $\mathrm{c} > 0$, then decreasing c we obtain a sequence of the Fourier transforms. The limit of this sequence, as $c \to 0$, is assumed to be the Fourier transform of $x(t)$. This transform is known under the name of the Fourier transform in a limit sense.

In Section 2 we defined the Laplace transform

$$\mathcal{L}(x(t)) = X(s) = \int_{0}^{\infty} x(t)\mathrm{e}^{-st}\mathrm{d}t$$

where $s = \sigma + \mathrm{j}\omega$ is a complex variable called the complex frequency. To find the relationship of the Laplace transform to the Fourier transform given by (9.8) we consider a signal $x(t)u(t)$. Then, we may write

$$\mathscr{F}\left(x(t)\,u(t)\right) = X\left(\mathrm{j}\omega\right) = \int_{-\infty}^{\infty} x(t)u(t)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t =$$

$$= \int_{0}^{\infty} x(t)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t = X\left(s\right)\Big|_{s=\mathrm{j}\omega} \quad .$$

Thus, in this case we can obtain the Fourier transform from the Laplace transform replacing s by jω

$$\mathscr{F}\left(x(t)\,u(t)\right) = \mathscr{L}\left(x(t)\right)\Big|_{s=\mathrm{j}\omega}$$

provided that each transform exists. For example if

$$x(t) = K\mathrm{e}^{-at}u(t), \qquad a > 0$$

then

$$X\left(s\right) = \frac{K}{s+a}$$

and

$$X\left(\mathrm{j}\omega\right) = \frac{K}{\mathrm{j}\omega + a} \ .$$

Example 9.1

Let us consider a rectangular pulse shown in Fig.9.3.

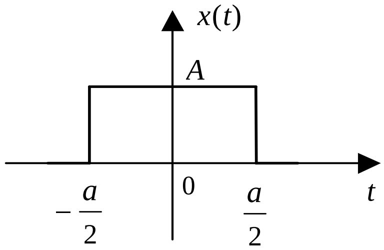

Fig. 9.3. A rectangular pulse

To find the Fourier transform of this signal, we use (9.8)

$$X(j\omega)=\int_{-\frac{a}{2}}^{\frac{a}{2}} A\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t=-\frac{A}{\mathrm{j}\omega}\mathrm{e}^{-\mathrm{j}\omega t}\Big|_{-\frac{a}{2}}^{\frac{a}{2}}=-\frac{A}{\mathrm{j}\omega}\left(\mathrm{e}^{-\mathrm{j}\omega\frac{a}{2}}-\mathrm{e}^{\mathrm{j}\omega\frac{a}{2}}\right)=$$

$$=\frac{2A}{\omega}\frac{\mathrm{e}^{\mathrm{j}\omega\frac{a}{2}}-\mathrm{e}^{-\mathrm{j}\omega\frac{a}{2}}}{2\mathrm{j}}=\frac{2A}{\omega}\sin\frac{\omega a}{2}=aA\frac{\sin\frac{1}{2}\omega a}{\frac{1}{2}\omega a}.$$

Thus, we have

$$X(\mathrm{j}\omega)=aA\frac{\sin\frac{1}{2}\omega a}{\frac{1}{2}\omega a}. \tag{9.10}$$

Observe that apart from the constant aA the function (9.10) has the form of the function $\operatorname{sinc} x=\frac{\sin x}{x}$ where $x=\frac{1}{2}\omega a$. The plot of this function is depicted in Fig.9.4.

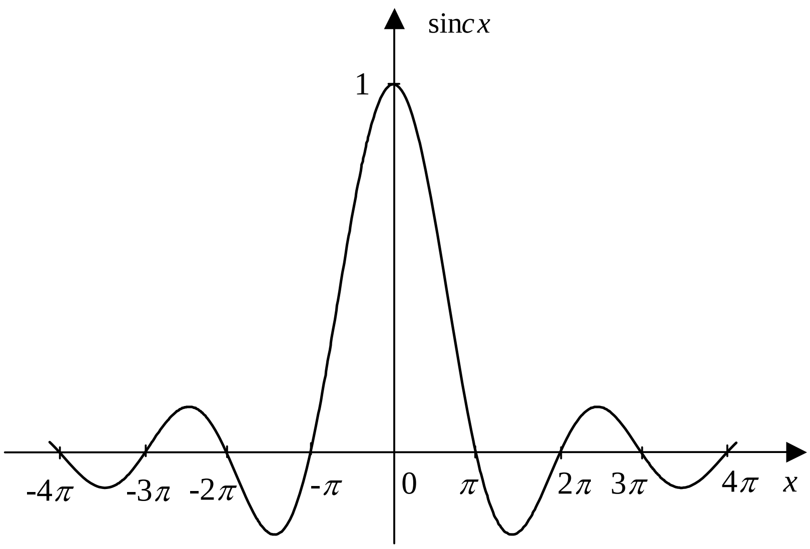

Fig. 9.4. Plot of $\operatorname{sinc} x$

Hence, the plot of $X(\mathrm{j}\omega)$ as a function of ω is as shown in Fig.9.5.

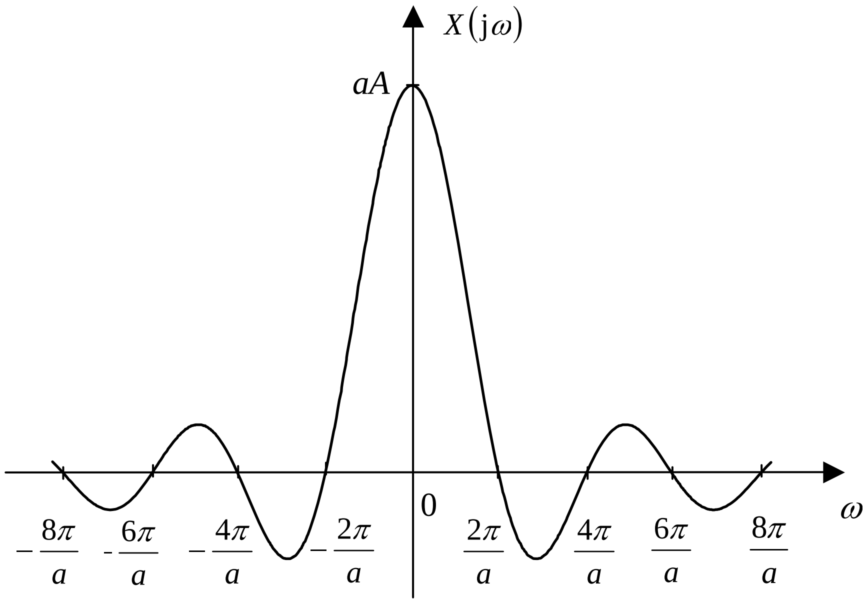

Fig. 9.5. Plot of the Fourier transform of the pulse shown in Fig.9.3

9.2. Amplitude and phase spectra

Let us consider a rectangular pulse train as shown in Fig.9.6.

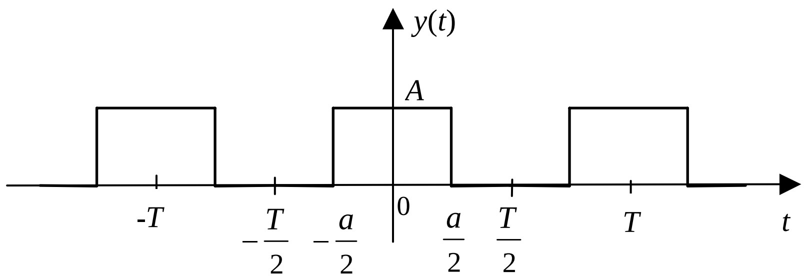

Fig. 9.6. Rectangular pulse train

The Fourier coefficients for this signal are given by

$$\tilde{c}_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} y(t)\mathrm{e}^{-\mathrm{j}n\omega_0 t}\mathrm{d}t = \frac{1}{T}\int_{-\frac{a}{2}}^{\frac{a}{2}} A\mathrm{e}^{-\mathrm{j}n\omega_0 t}\mathrm{d}t$$

where

$$\omega_0 = \frac{2\pi}{T}.$$

Hence, we have

$$\tilde{c}_n = \frac{Aa}{T}\frac{\sin\frac{1}{2}n\omega_0 a}{\frac{1}{2}n\omega_0 a}$$

or

$$\tilde{c}_n T = Aa\frac{\sin\frac{1}{2}n\omega_0 a}{\frac{1}{2}n\omega_0 a}. \qquad (9.11)$$

Let us compare the Fourier transform $X(\mathrm{j}\omega)$ of the rectangular pulse shown in Fig.9.3 (see (9.10)) and $\tilde{c}_n T$ for the rectangular pulse train shown in Fig.9.6 (see (9.11)). Both relationships have the same form, however, $X(\mathrm{j}\omega)$ is a function of the continuous variable ω, whereas $\tilde{c}_n T$ has values for discrete frequencies $n\omega_0$. Hence, the magnitude of $X(\mathrm{j}\omega)$ is the envelope of the magnitudes of $\tilde{c}_n T$.

We examine $\tilde{c}_n T$ as *a* is fixed and *T* changes and assumes three values: 2*a*, 4*a*, and 8*a*.

The envelope for $|\tilde{c}_n T|$ does not depend on *T*; consequently it is the same in the three cases and given by

$$|X(\mathrm{j}\omega)| = Aa\left|\frac{\sin\frac{1}{2}\omega a}{\frac{1}{2}\omega a}\right|.$$

The plot of the envelope is shown in Fig.9.7.

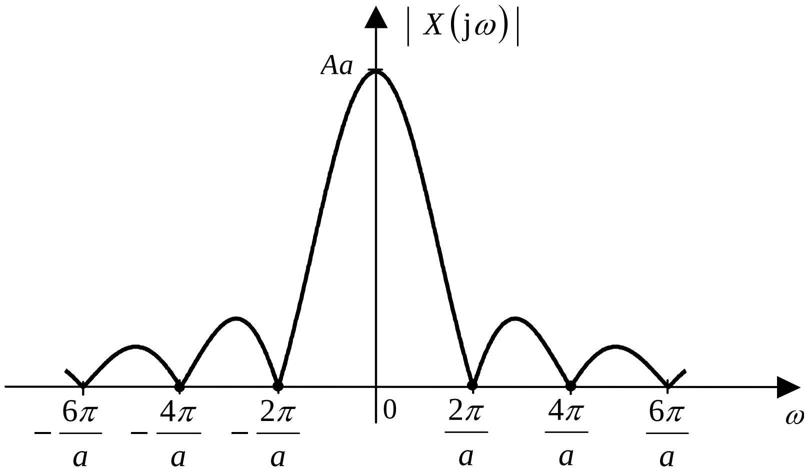

Fig. 9.7. Plot of the envelope for $|\tilde{c}_n T|$

For $T = 2a$ the amplitude spectrum $|\tilde{c}_n T|$ is defined at the frequencies

$$n\omega_0 = n\frac{2\pi}{T} = n\frac{2\pi}{2a} = n\frac{\pi}{a}.$$

This spectrum is depicted in Fig.9.8

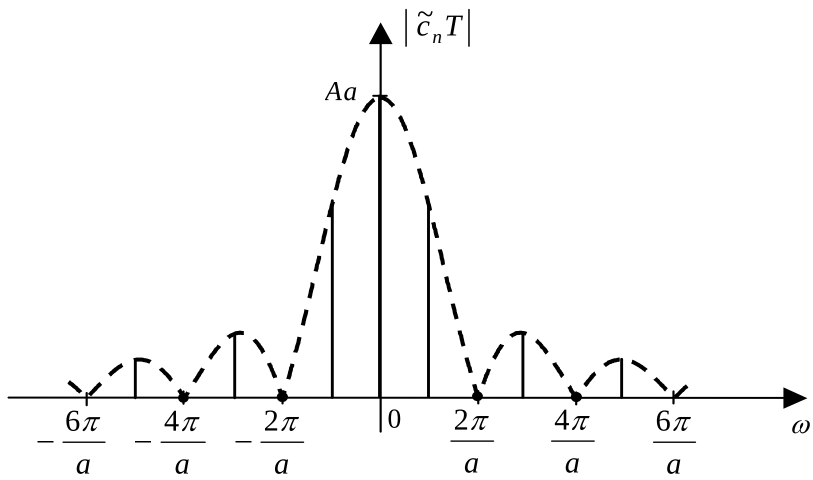

Fig. 9.8. The amplitude spectrum $|\tilde{c}_n T|$ for $T = 2a$

Now we consider the case where $T = 4a$. Then the amplitude spectrum $|\tilde{c}_n T|$ is defined at frequencies

$$n\omega_0 = n\frac{2\pi}{T} = n\frac{2\pi}{4a} = n\frac{\pi}{2a} .$$

This spectrum is depicted in Fig.9.9.

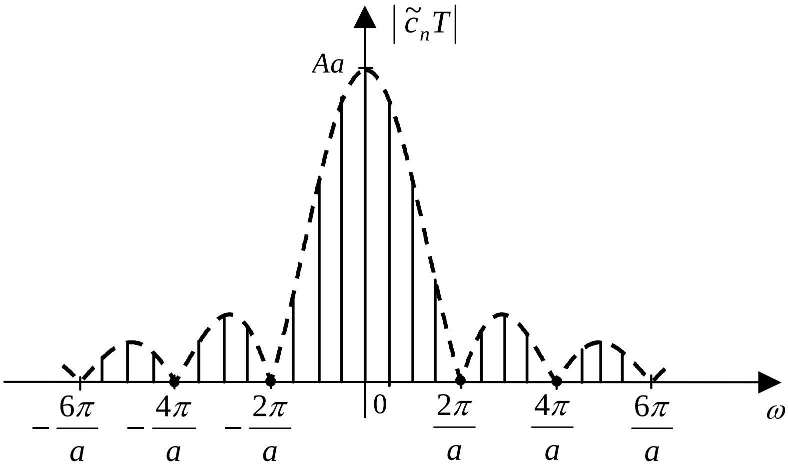

Fig. 9.9. The amplitude spectrum $|\tilde{c}_n T|$ for $T = 4a$

For $T = 8a$ the amplitude spectrum $|\tilde{c}_n T|$ is defined at frequencies

$$n\omega_0 = n\frac{2\pi}{T} = n\frac{2\pi}{8a} = n\frac{\pi}{4a} .$$

The spectrum is shown in Fig.9.10.

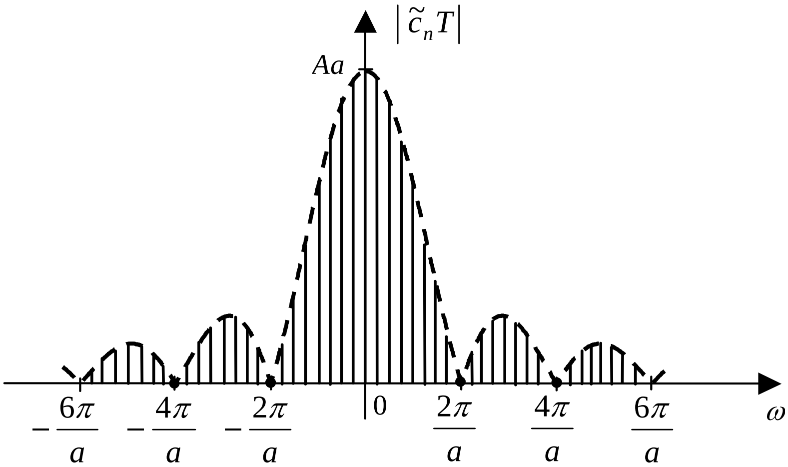

Fig. 9.10. The amplitude spectrum $|\tilde{c}_n T|$ for $T = 8a$

The above discussion shows that when T increases, the separation of the amplitude spectrum lines of $\tilde{c}_n T$ decreases. Thus, the number of harmonics in a given frequency interval increases. In the limit as $T \to \infty$ the distance between the lines tends to zero. Consequently, the discrete spectrum $|\tilde{c}_n T|$ becomes the continuous spectrum represented by the envelope of $|\tilde{c}_n T|$, i.e. by $|X(\mathrm{j}\omega)|$.

Another interpretation is as follows. As T increases the distance between frequencies of subsequent harmonics decreases. In the limit as $T \to \infty$ this distance tends to zero and the amplitudes of the exponential frequencies represented by $|\tilde{c}_n|$ tend to zero, because

$$|\tilde{c}_n| = \frac{|\tilde{c}_n T|}{T}$$

and $|\tilde{c}_n T|$ is framed by $|X(\mathrm{j}\omega)|$.

The above interpretation has been given for the rectangular pulse but it is valid in general. For an arbitrary signal $x(t)$, defined in a finite time interval $\left[-\frac{a}{2}, \frac{a}{2}\right]$, and equal to zero outside this interval, we create a periodic signal $y(t)$ with period $T > a$ which is identical with $x(t)$ for $-\frac{T}{2} < t < \frac{T}{2}$. The exponential Fourier series coefficients $\tilde{c}_n$ of the signal $y(t)$ satisfy the relation

$$\tilde{c}_n T = \int_{-\frac{a}{2}}^{\frac{a}{2}} x(t) \mathrm{e}^{-\mathrm{j}n\omega_0 t} \mathrm{d}t \, . \tag{9.12}$$

The Fourier transform of $x(t)$ is

$$X(j\omega) = \int_{-\frac{a}{2}}^{\frac{a}{2}} x(t) \mathrm{e}^{-\mathrm{j}\omega t} \mathrm{d}t \, . \tag{9.13}$$

Relationships (9.12) and (9.13) are identical in form but the former is defined for discrete frequencies whereas the latter for continuous frequencies. As a matter of fact, $\tilde{c}_n T$ equals the samples of $X(\mathrm{j}\omega)$ at

$$\omega = n\omega_0 = n\frac{2\pi}{T} \, .$$

The magnitude $\left|X(\mathrm{j}\omega)\right|$ will be called the continuous amplitude spectrum and $\angle\left(X(\mathrm{j}\omega)\right)$ the continuous phase spectrum of a nonperiodic signal $x(t)$.

9.3. Properties of the Fourier transform

In this Section we consider a number of properties of the Fourier transform. Recall that a signal $x(t)$ and its Fourier transform $X(\mathrm{j}\omega)$ are related by the pair of equations:

$$\mathscr{F}(x(t)) = X(\mathrm{j}\omega) = \int_{-\infty}^{\infty} x(t)\,\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t \tag{9.14}$$

$$\mathscr{F}^{-1}(X(\mathrm{j}\omega)) = x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\mathrm{j}\omega)\,\mathrm{e}^{\mathrm{j}\omega t}\mathrm{d}\omega\,. \tag{9.15}$$

Linearity

If $X_1(\mathrm{j}\omega)$ and $X_2(\mathrm{j}\omega)$ are the Fourier transforms of $x_1(t)$ and $x_2(t)$, respectively, then for arbitrary constants c_1 and c_2 (real or complex) the Fourier transform of the sum:

$$c_1 x_1(t) + c_2 x_2(t) \tag{9.16}$$

is

$$c_1 X_1(\mathrm{j}\omega) + c_2 X_2(\mathrm{j}\omega). \tag{9.17}$$

This property follows directly by substituting (9.16) into (9.14).

Scaling

If $x(t)$ has the Fourier transform $X(\mathrm{j}\omega)$, then for any real constant $\alpha \neq 0$

$$\mathscr{F}(x(\alpha t)) = \frac{1}{|\alpha|} X\left(\mathrm{j}\frac{\omega}{\alpha}\right). \tag{9.18}$$

Proof

To prove the property we consider two cases of positive and negative values of α.

For $\alpha > 0$ let $u = \alpha t$. Substituting this into (9.14) yields

$$\mathscr{F}\big(x(\alpha t)\big) = \int_{-\infty}^{\infty} x(\alpha t)\,\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t = \frac{1}{\alpha}\int_{-\infty}^{\infty} x(u)\,\mathrm{e}^{-\mathrm{j}\frac{\omega}{\alpha}u}\mathrm{d}u = \frac{1}{|\alpha|}X\left(\mathrm{j}\frac{\omega}{\alpha}\right).$$

For $\alpha < 0$ we have

$$\mathscr{F}\big(x(\alpha t)\big) = \int_{-\infty}^{\infty} x(\alpha t)\,\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t = \frac{-1}{\alpha}\int_{\infty}^{-\infty} x(u)\,\mathrm{e}^{-\mathrm{j}\frac{\omega}{\alpha}u}\mathrm{d}u = \frac{1}{|\alpha|}X\left(\mathrm{j}\frac{\omega}{\alpha}\right).$$

Combining these two results gives the relationship (9.18).

Letting $\alpha = -1$, we obtain

$$\mathscr{F}\big(x(-t)\big) = X(-\mathrm{j}\omega). \tag{9.19}$$

Thus, reversing a signal in time implicates reversing its Fourier transform. The scaling property states that scaling in time by a factor α corresponds to scaling in frequency by a factor $\frac{1}{\alpha}$. Thus, when the time scale is expanded by the factor, the frequency spectrum is contracted by the same factor and vice versa. It is illustrated in figures 9.11 and 9.12.

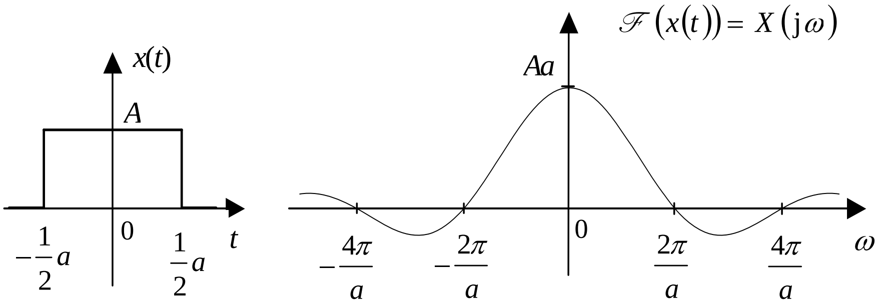

Fig. 9.11. Rectangular pulse and its frequency spectrum

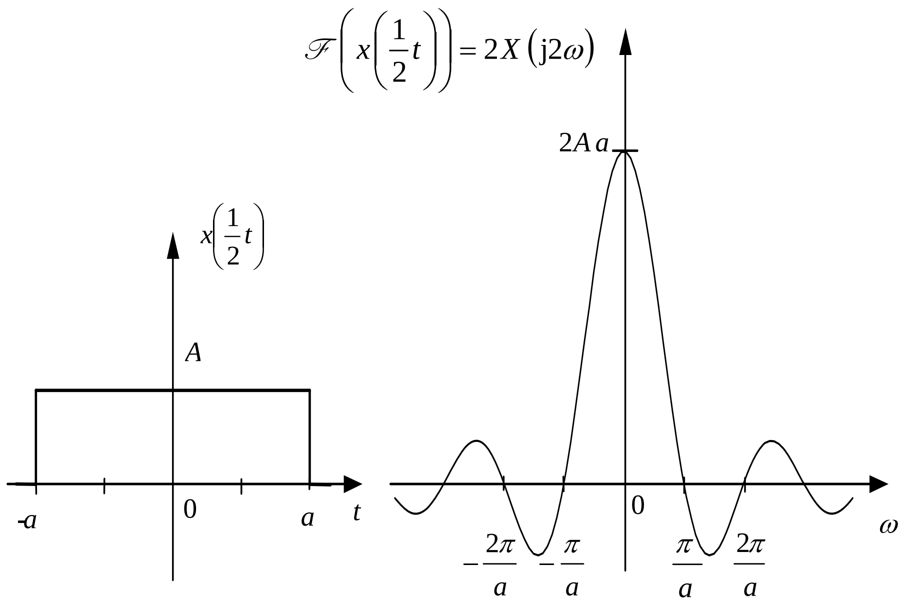

Fig. 9.12. Rectangular pulse with expended time scale and its frequency spectrum

Let us consider a situation where $x(t)$ is a real function of *t*. Using Euler's identity we obtain

$$X\left(\mathrm{j}\omega\right) = \int_{-\infty}^{\infty} x(t)\left(\cos\omega t - \mathrm{j}\sin\omega t\right)\mathrm{d}t =$$

$$= \int_{-\infty}^{\infty} x(t)\cos\omega t\,\mathrm{d}t - \mathrm{j}\int_{-\infty}^{\infty} x(t)\sin\omega t\,\mathrm{d}t\ .$$

Letting $\int_{-\infty}^{\infty} x(t)\cos\omega t\,\mathrm{d}t = U(\omega)$ and $\int_{-\infty}^{\infty} x(t)\sin\omega t\,\mathrm{d}t = V(\omega)$, we obtain

$$X\left(j\omega\right) = U(\omega) - \mathrm{j}V(\omega).$$

Since $U(\omega)$ is even, i.e. $U(-\omega) = U(\omega)$ and $V(\omega)$ is odd, i.e. $V(-\omega) = -V(\omega)$, then

$$X(-j\omega)=U(\omega)+\mathrm{j}V(\omega).$$

Hence, we have

$$|X(\mathrm{j}\omega)|=|X(-\mathrm{j}\omega)| \qquad \angle X(\mathrm{j}\omega)=-\angle X(-\mathrm{j}\omega).$$

Thus, the magnitude spectrum is even and the phase spectrum is odd.

Example 9.2

Given a fixed positive number α, let $x(t)$ denote the signal defined by

$$x(t)=\begin{cases} A\cos\omega_0 t, & -\alpha \le t \le \alpha \\ 0, & \text{all other } t \end{cases}.$$

Plot of $x(t)$ is shown in Fig. 9.13.

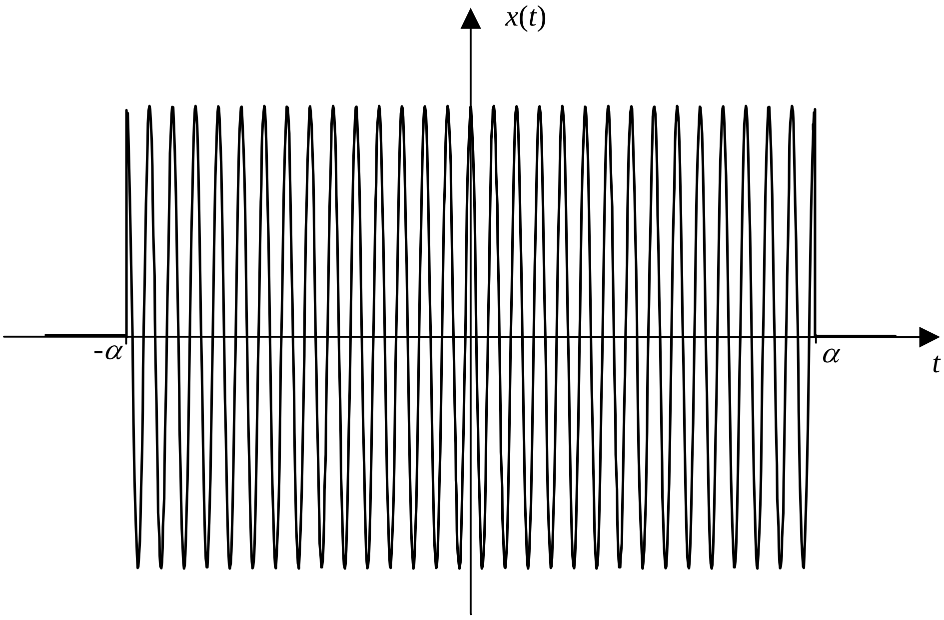

Fig. 9.13. Plot of the signal in Example 9.2

Its Fourier transform is

$$X(\mathrm{j}\omega)=\int_{-\infty}^{\infty}x(t)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t=A\int_{-\alpha}^{\alpha}\cos\omega_0 t\cos\omega t\,\mathrm{d}t-\mathrm{j}A\int_{-\alpha}^{\alpha}\cos\omega_0 t\sin\omega t\,\mathrm{d}t=$$

$$=2A\int_{0}^{\alpha}\cos\omega_0 t\cos\omega t\,\mathrm{d}t=A\int_{0}^{\alpha}\left(\cos(\omega-\omega_0)t+\cos(\omega+\omega_0)t\right)\mathrm{d}t=$$

$$=A\alpha\left(\frac{\sin(\omega-\omega_0)\alpha}{(\omega-\omega_0)\alpha}+\frac{\sin(\omega+\omega_0)\alpha}{(\omega+\omega_0)\alpha}\right)\quad .$$

The spectrum of $x(t)$ is depicted in Fig. 9.14.

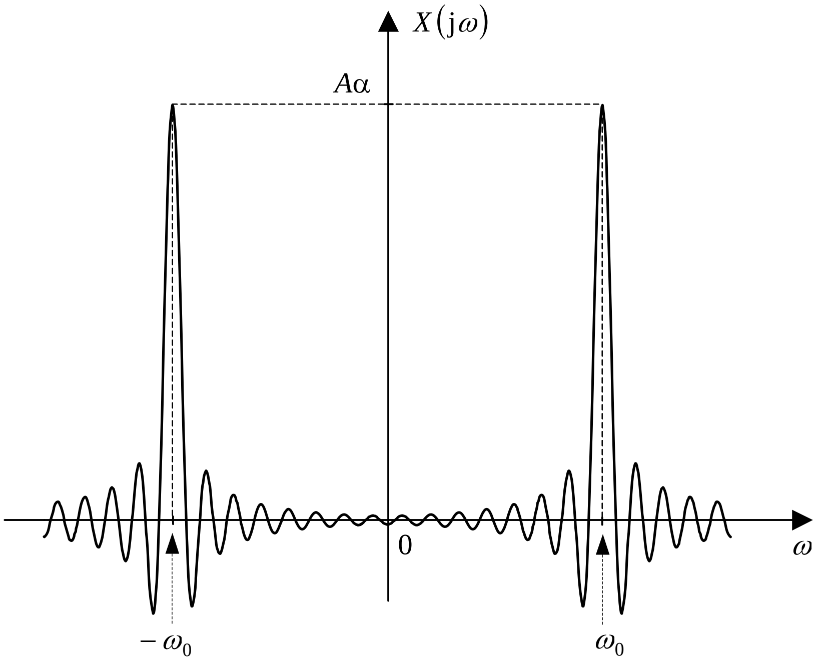

Fig. 9.14. Plot of the spectrum of the signal in Example 9.2

From this figure it is seen that most of the spectral content of the signal is concentrated in the neighborhoods of ω_0 and $(-\omega_0)$.

Time shifting

Let us consider a time signal $x(t)$ delayed by t_0 i.e. $x(t-t_0)$.
If

$$\mathscr{F}(x(t))=X(\mathrm{j}\omega)$$

then

$$\mathscr{F}\left(x(t-t_0)\right)=\mathrm{e}^{-\mathrm{j}\omega t_0}X\left(\mathrm{j}\omega\right). \tag{9.20}$$

Proof

We set $x(t-t_0)$ into the definition formula

$$\mathscr{F}\left(x(t-t_0)\right)=\int_{-\infty}^{\infty}x(t-t_0)\,\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t=\int_{-\infty}^{\infty}x(t-t_0)\,\mathrm{e}^{-\mathrm{j}\omega(t-t_0)}\mathrm{e}^{-\mathrm{j}\omega t_0}\mathrm{d}(t-t_0).$$

Letting $u=t-t_0$, we have

$$\mathscr{F}\left(x(t-t_0)\right)=\mathrm{e}^{-\mathrm{j}\omega t_0}\int_{-\infty}^{\infty}x(u)\,\mathrm{e}^{-\mathrm{j}\omega u}\mathrm{d}u=\mathrm{e}^{-\mathrm{j}\omega t_0}X\left(\mathrm{j}\omega\right).$$

This property states that if a signal is delayed by t_0, its Fourier transform equals the Fourier transform of the original signal multiplied by $\mathrm{e}^{-\mathrm{j}\omega t_0}$. Consequently, the amplitude spectrum is not affected because

$$\left|\mathrm{e}^{-\mathrm{j}\omega t_0}X\left(\mathrm{j}\omega\right)\right|=\left|\mathrm{e}^{-\mathrm{j}\omega t_0}\right|\left|X\left(\mathrm{j}\omega\right)\right|=\left|X\left(\mathrm{j}\omega\right)\right|$$

whereas the phase spectrum is shifted by $-\omega t_0$

$$\angle\left(\mathrm{e}^{-\mathrm{j}\omega t_0}X(\mathrm{j}\omega)\right)=\angle\left(X(\mathrm{j}\omega)\right)-\omega t_0.$$

Frequency shifting. Modulation

If $x(t)$ has the Fourier transform $X\left(\mathrm{j}\omega\right)$, then

$$\mathscr{F}\left(x(t)\,\mathrm{e}^{\mathrm{j}\omega_0 t}\right)=X\left(\mathrm{j}(\omega-\omega_0)\right). \tag{9.21}$$

Proof

We substitute $x(t)\mathrm{e}^{\mathrm{j}\omega_0 t}$ into (9.14) and rearrange as follows

$$\mathscr{F}\left(x(t)\,\mathrm{e}^{\mathrm{j}\omega_0 t}\right)=\int_{-\infty}^{\infty}x(t)\,\mathrm{e}^{\mathrm{j}\omega_0 t}\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t=\int_{-\infty}^{\infty}x(t)\,\mathrm{e}^{-\mathrm{j}(\omega-\omega_0)t}\mathrm{d}t=X\left(\mathrm{j}(\omega-\omega_0)\right).$$

Example 9.3

Let us consider the signal

$$g(t) = x(t)\cos\omega_0 t\ . \qquad (9.22)$$

The signal $x(t)$ is called the modulating signal and $\cos\omega_0 t$ is called the carrier or the modulated signal.
Since

$$\cos\omega_0 t = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

then

$$g(t) = \frac{1}{2}x(t)\,e^{j\omega_0 t} + \frac{1}{2}x(t)e^{-j\omega_0 t}\ .$$

Using the frequency shifting property, we obtain

$$\mathscr{F}(g(t)) = \frac{1}{2}X\left(j(\omega - \omega_0)\right) + \frac{1}{2}X\left(j(\omega + \omega_0)\right). \qquad (9.23)$$

The above equation states that by multiplying the time function $x(t)$ by $\cos\omega_0 t$ the original spectrum $X(j\omega)$ is split into two parts so that half of it is shifted by ω_0 and the other half is shifted by $-\omega_0$ (see Figs.9.15 and .9.16).

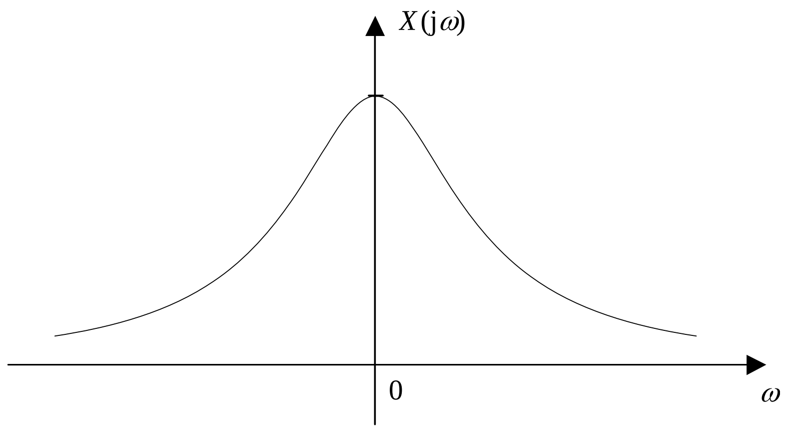

Fig. 9.15. Spectrum of an example signal $x(t)$

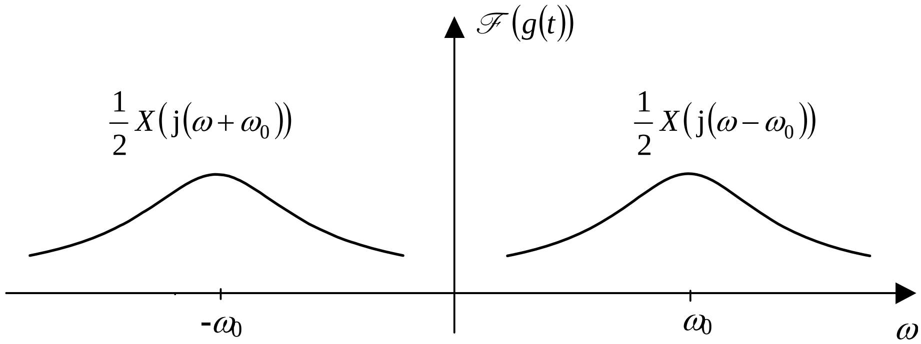

Fig. 9.16. Spectrum of the signal $g(t)=x(t)\cos\omega_0 t$

Differentiation

Let $x(t)$ be a signal with Fourier transform $X(j\omega)$; then

$$\mathscr{F}\left(\frac{dx(t)}{dt}\right)=j\omega X(j\omega). \tag{9.24}$$

Proof

We use the inverse Fourier transform

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}d\omega$$

and differentiate with respect to t

$$\frac{dx(t)}{dt}=\frac{1}{2\pi}\int_{-\infty}^{\infty}j\omega X(j\omega)e^{j\omega t}d\omega .$$

We recognize the expression on the right hand side as the inverse Fourier transform of $j\omega X(j\omega)$. Thus, the equation

$$\mathscr{F}\left(\frac{dx(t)}{dt}\right)=j\omega X(j\omega)$$

holds. This result can be directly extended to the *n*-th derivative by repeated differentiations

$$\frac{\mathrm{d}^n x(t)}{\mathrm{d}t^n} = (\mathrm{j}\omega)^n X(\mathrm{j}\omega). \tag{9.25}$$

Example 9.4

Let us consider the triangular pulse shown in Fig.9.17.

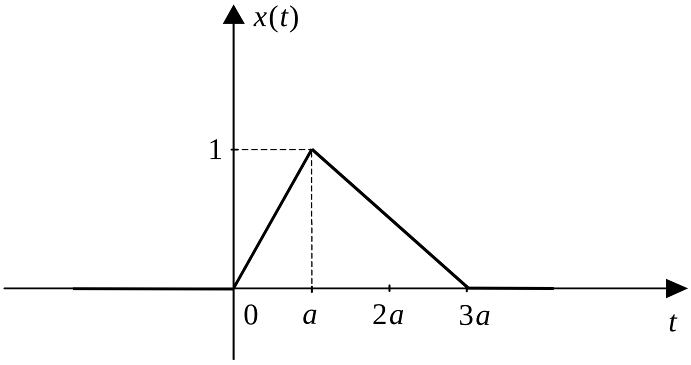

Fig. 9.17. Triangular pulse signal

The derivative of $x(t)$ is a rectangular pulse signal (see Fig.9.18)

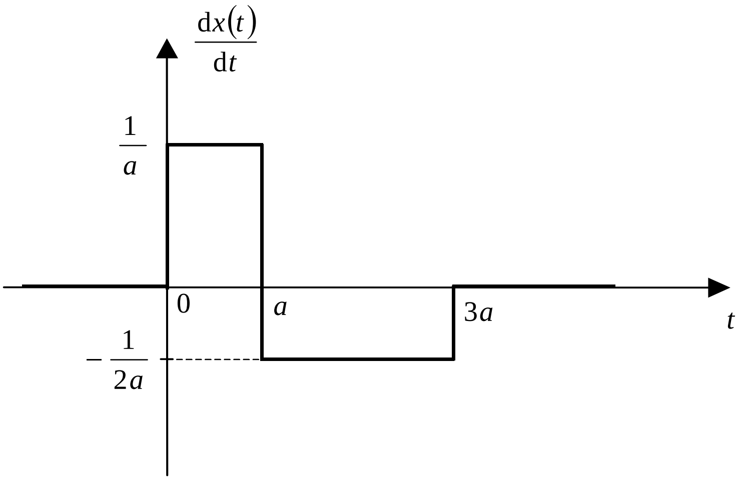

Fig. 9.18. Derivative of the triangular signal of Figure 9.17

The Fourier transform of $\frac{\mathrm{d}x}{\mathrm{d}t}$ is

$$\mathscr{F}\left(\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right)=\frac{1}{a}\int_0^a \mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t-\frac{1}{2a}\int_a^{3a}\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t=\frac{1}{a}\frac{1}{\mathrm{j}\omega}\left(-\mathrm{e}^{-\mathrm{j}\omega t}\Big|_0^a+\frac{1}{2}\mathrm{e}^{-\mathrm{j}\omega t}\Big|_a^{3a}\right)=$$

$$=\frac{1}{\mathrm{j}\omega a}\left(1-\mathrm{e}^{-\mathrm{j}\omega a}+\frac{1}{2}(\mathrm{e}^{-\mathrm{j}\omega 3a}-\mathrm{e}^{-\mathrm{j}\omega a})\right)=$$

$$=\frac{1}{\mathrm{j}\omega a}\left(1-\frac{3}{2}\mathrm{e}^{-\mathrm{j}\omega a}+\frac{1}{2}\mathrm{e}^{-\mathrm{j}\omega 3a}\right).$$

Application of the differentiation property yields

$$\mathrm{j}\omega X(\mathrm{j}\omega)=\frac{1}{\mathrm{j}\omega a}\left(1-\frac{3}{2}\mathrm{e}^{-\mathrm{j}\omega a}+\frac{1}{2}\mathrm{e}^{-\mathrm{j}\omega 3a}\right).$$

Hence, the equation

$$X(\mathrm{j}\omega)=\frac{1}{\omega^2 a}\left(\frac{3}{2}\mathrm{e}^{-\mathrm{j}\omega a}-\frac{1}{2}\mathrm{e}^{-\mathrm{j}\omega 3a}-1\right)$$

holds.

Conjugation property

The conjugation property states that

$$\mathscr{F}\left(x^*(t)\right)=X^*(-\mathrm{j}\omega) \tag{9.26}$$

where $X(\mathrm{j}\omega)$ is the Fourier transform of the signal $x(t)$ which is generally complex and * is a symbol of conjugation.

Proof

Since

$$X(\mathrm{j}\omega)=\int_{-\infty}^{\infty}x(t)\,\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t$$

then

$$X^*(\mathrm{j}\omega)=\int_{-\infty}^{\infty}x^*(t)\,\mathrm{e}^{\mathrm{j}\omega t}\mathrm{d}t\,.$$

Replacing ω by -ω we obtain

$$X^*(-\mathrm{j}\omega)=\int_{-\infty}^{\infty} x^*(t)\,\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t=\mathscr{F}\left(x^*(t)\right).$$

In a special case where $x(t)$ is real, $x^*(t)=x(t)$ holds and

$$X^*(-\mathrm{j}\omega)=\int_{-\infty}^{\infty} x(t)\,\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t=X(\mathrm{j}\omega).$$

9.4. Convolution

Let us consider signals $x_1(t)$ and $x_2(t)$ having the Fourier transforms $X_1(\mathrm{j}\omega)$ and $X_2(\mathrm{j}\omega)$, respectively. The convolution of $x_1(t)$ and $x_2(t)$ is

$$\mathrm{x}_1(t)*x_2(t)=\int_{-\infty}^{\infty} x_1(\tau)\,x_2(t-\tau)\mathrm{d}\tau\ .$$

The following theorem, called the convolution theorem, holds

$$\mathscr{F}\left(x_1(t)*x_2(t)\right)=X_1(\mathrm{j}\omega)X_2(\mathrm{j}\omega). \tag{9.27}$$

Example 9.5

Figure 9.19 shows signals $x_1(t)$ and $x_2(t)$.

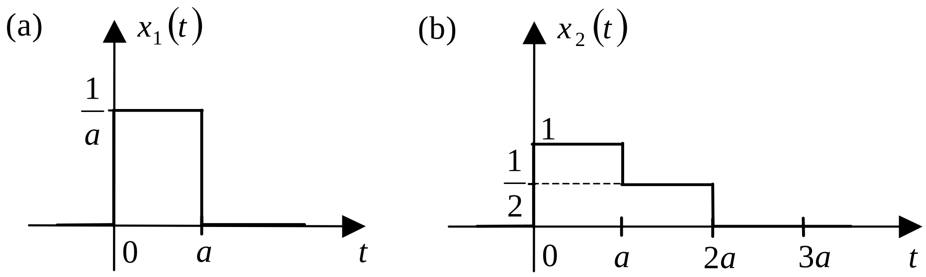

Fig. 9.19. Signals for Example 9.5

We determine the convolution of these signals

$$f(t) = x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, \mathrm{d}\tau \, .$$

Using the graphical approach we find the convolution as shown in Fig.9.20

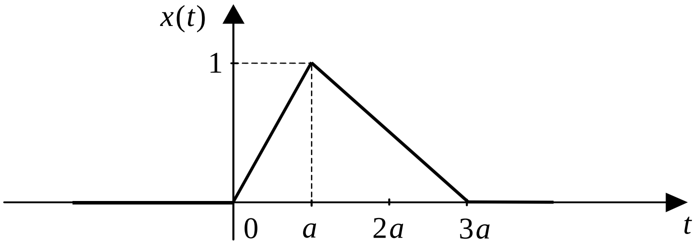

Fig. 9.20. Convolution of the signals shown in Figure 9.19

The Fourier transforms of $x_1(t)$ and $x_2(t)$ are:

$$X_1(\mathrm{j}\omega) = \frac{1}{a}\int_0^a \mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t = \frac{1}{\mathrm{j}\omega a}\left(1-\mathrm{e}^{-\mathrm{j}\omega a}\right)$$

$$X_2(\mathrm{j}\omega) = \int_0^a \mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t + \frac{1}{2}\int_a^{2a} \mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t = \frac{1}{\mathrm{j}\omega}\left(1 - \frac{1}{2}(\mathrm{e}^{-\mathrm{j}\omega a} + \mathrm{e}^{-\mathrm{j}\omega 2a})\right).$$

Thus, using the convolution theorem, we obtain

$$\mathscr{F}(f(t)) = X_1(\mathrm{j}\omega) X_2(\mathrm{j}\omega) = \frac{1}{\omega^2 a}\left(\frac{3}{2}\mathrm{e}^{-\mathrm{j}\omega a} - \frac{1}{2}\mathrm{e}^{-\mathrm{j}\omega 3a} - 1\right).$$

Now we find the Fourier transform of the product of two signals. Let $X_1(\mathrm{j}\omega) = \mathscr{F}(x_1(t))$ and $X_2(\mathrm{j}\omega) = \mathscr{F}(x_2(t))$, then

$$\mathscr{F}(x_1(t)x_2(t)) = \int_{-\infty}^{\infty} x_1(t) x_2(t) \mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t =$$

$$= \int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\mathrm{j}r)\mathrm{e}^{\mathrm{j}rt}\mathrm{d}r\right) x_2(t)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\mathrm{j}r)\left(\int_{-\infty}^{\infty} x_2(t)\mathrm{e}^{-\mathrm{j}(\omega - r)t}\mathrm{d}t\right)\mathrm{d}r = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\mathrm{j}r) X_2(\mathrm{j}(\omega - r))\mathrm{d}r \, .$$

The integral on the right hand side is called the frequency convolution and denoted by

$$\int_{-\infty}^{\infty} X_1(\mathrm{j}r) X_2(\mathrm{j}(\omega - r))\mathrm{d}r = X_1(\mathrm{j}r) * X_2(\mathrm{j}r).$$

Thus, it holds

$$\mathscr{F}(x_1(t)x_2(t)) = \frac{1}{2\pi} X_1(\mathrm{j}\omega) * X_2(\mathrm{j}\omega). \tag{9.28}$$

Duality

Recall the inverse Fourier transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\mathrm{j}\omega) \mathrm{e}^{\mathrm{j}\omega t} \mathrm{d}\omega$$

and change the sign of t as well as multiply both sides by 2π

$$2\pi x(-t) = \int_{-\infty}^{\infty} X(\mathrm{j}\omega) \mathrm{e}^{-\mathrm{j}\omega t} \mathrm{d}\omega .$$

Let us interchange variables ω and t

$$\omega \leftrightarrow t \qquad \text{or} \qquad \mathrm{j}\omega \leftrightarrow \mathrm{j}t$$

$$2\pi x(-\omega) = \int_{-\infty}^{\infty} X(\mathrm{j}t) \mathrm{e}^{-\mathrm{j}\omega t} \mathrm{d}t .$$

Next we label $\hat{x}(t) = X(\mathrm{j}t)$ finding

$$2\pi x(-\omega) = \int_{-\infty}^{\infty} \hat{x}(t) \mathrm{e}^{-\mathrm{j}\omega t} \mathrm{d}t = \mathscr{F}(\hat{x}(t)). \tag{9.29}$$

Equation (9.29) shows the duality property of the Fourier transform which enables us to express the Fourier transform of the signal $\hat{x}(t)$ having the same form as $X(\mathrm{j}\omega)$ in terms of $x(-\omega)$ having the same form as $x(-t)$.

Example 9.6

The signal $x(t)$ shown in Fig.9.19 has the Fourier transform

$$X(j\omega) = a\left(\frac{\sin\frac{1}{2}a\omega}{\frac{1}{2}a\omega}\right)^2 .$$

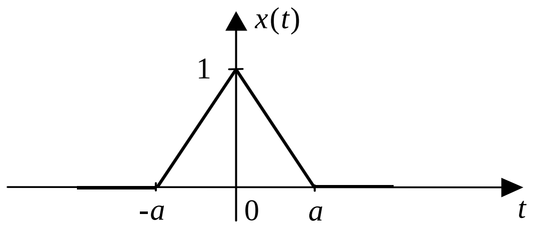

Fig. 9.21. Triangular signal for Example 9.6

Hence, the signal $\hat{x}(t) = X(jt)$ is

$$\hat{x}(t) = a\left(\frac{\sin\frac{1}{2}at}{\frac{1}{2}at}\right)^2 .$$

Using the duality property, we obtain

$$\mathscr{F}(\hat{x}(t)) = 2\pi x(-\omega) = \tilde{X}(j\omega)$$

where $\tilde{X}(j\omega)$ is shown in Fig.9.20.

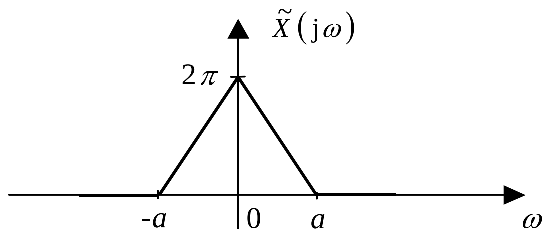

Fig. 9.22. Fourier transform of $\hat{x}(t)$

9.5. Generalized Fourier transform

The Fourier transform of the unit impulse $\delta(t)$ defined in Section 1 is

$$\mathscr{F}\left(\delta(t)\right)=\int\limits_{-\infty}^{\infty}\delta(t)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t=\mathrm{e}^{-\mathrm{j}\omega t}\Big|_{t=0}=1$$

or

$$\mathscr{F}\left(\delta(t)\right)=1\,. \tag{9.30}$$

Let us consider the shifted unit impulse $\delta(t-t_0)$. Its Fourier transform can be found using the time-shifting property

$$\mathscr{F}\left(\delta(t-t_0)\right)=\mathrm{e}^{-\mathrm{j}\omega t_0}\ . \tag{9.31}$$

Now we find a time function $f(t)$ such that its Fourier transform is $\delta(\omega)$. We use the inverse Fourier transform

$$f(t)=\mathscr{F}^{-1}\left(\delta(\omega)\right)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\delta(\omega)\,\mathrm{e}^{\mathrm{j}\omega t}\mathrm{d}\omega=\frac{1}{2\pi}\mathrm{e}^{\mathrm{j}\omega t}\Big|_{\omega=0}=\frac{1}{2\pi}\,.$$

Thus,

$$\mathscr{F}\left(\frac{1}{2\pi}\right)=\delta(\omega)$$

or

$$\mathscr{F}(1)=2\pi\delta(\omega) \tag{9.32}$$

holds.
Equation (9.32) says that the Fourier transform of the constant 1 is a unit impulse at the origin with the strength 2π.

If the Fourier transform of a time function is $\delta(\omega+\omega_0)$, then the function is

$$\mathscr{F}^{-1}\left(\delta(\omega+\omega_0)\right)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\delta(\omega+\omega_0)\,\mathrm{e}^{\mathrm{j}\omega t}\mathrm{d}\omega=\frac{1}{2\pi}\mathrm{e}^{\mathrm{j}\omega t}\Big|_{\omega=-\omega_0}=\frac{1}{2\pi}\mathrm{e}^{-j\omega_0 t}\ .$$

Hence, we have

$$\mathscr{F}\left(e^{-j\omega_0 t}\right) = 2\pi\delta(\omega + \omega_0) \tag{9.33}$$

and replacing ω_0 by $-\omega_0$, we obtain

$$\mathscr{F}\left(e^{j\omega_0 t}\right) = 2\pi\delta(\omega - \omega_0). \tag{9.34}$$

The above results enable us to find the Fourier transform of $\cos\omega_0 t$. Since

$$\cos\omega_0 t = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

by the linearity property we obtain

$$\mathscr{F}(\cos\omega_0 t) = \pi\left(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right). \tag{9.35}$$

Equation (9.35) is illustrated in Fig.9.23.

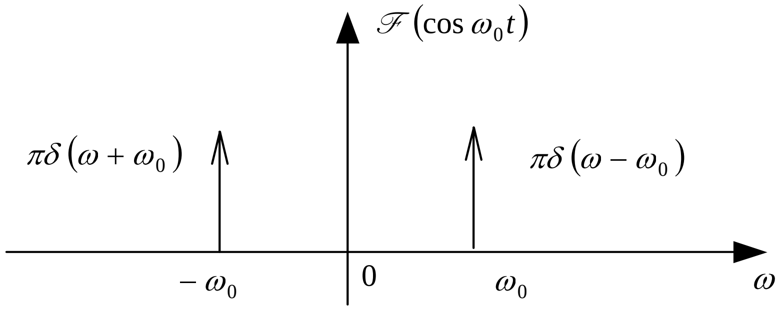

Fig. 9.23. Fourier transform of $\cos\omega_0 t$

Similarly we find the Fourier transform of $\sin\omega_0 t$

$$\mathscr{F}(\sin\omega_0 t) = \mathscr{F}\left(\frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)\right) = \frac{\pi}{j}\left(\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\right) =$$
$$= j\pi\left(\delta(\omega + \omega_0) - \delta(\omega - \omega_0)\right)$$

or

$$\mathrm{Re}\left(\mathscr{F}(\sin\omega_0 t)\right) = 0$$

$$Im(\mathscr{F}(\sin\omega_0 t)) = \pi(\delta(\omega+\omega_0) - \delta(\omega-\omega_0)). \tag{9.36}$$

Equation (9.36) is illustrated in Fig.9.24.

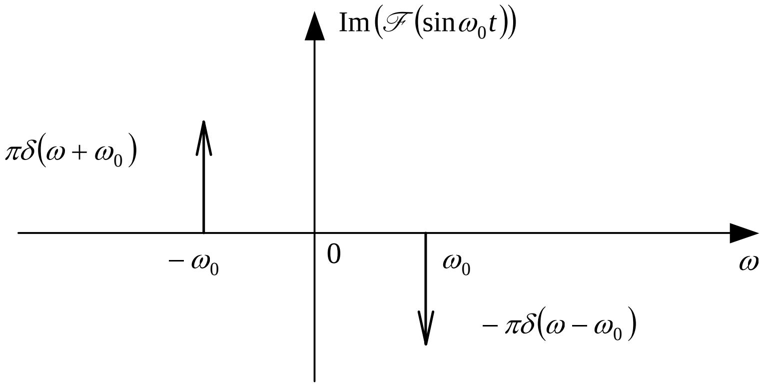

Fig. 9.24. Fourier transform of $\sin\omega_0 t$

9.6. The Fourier transform for periodic signals

Let us consider a periodic signal $x(t)$ with period T satisfying the Dirichlet conditions. The exponential Fourier series expansion of this signal is

$$x(t) = \sum_{k=-\infty}^{\infty} \tilde{c}_k e^{jk\omega_0 t} \tag{9.37}$$

where $\omega_0 = \dfrac{2\pi}{T}$. To find the Fourier transform of $x(t)$, we substitute (9.37) into the definition formula (9.14)

$$X(j\omega) = \int_{-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} \tilde{c}_k e^{jk\omega_0 t} \right) e^{-j\omega t} dt .$$

Changing the order of summing and integrating yields

$$X(j\omega) = \sum_{k=-\infty}^{\infty} \left(\tilde{c}_k \int_{-\infty}^{\infty} e^{jk\omega_0 t} e^{-j\omega t} dt \right) = \sum_{k=-\infty}^{\infty} \tilde{c}_k \mathscr{F}\left(e^{jk\omega_0 t}\right).$$

Since

$$\mathscr{F}\left(e^{jk\omega_0 t}\right) = 2\pi\delta\left(\omega - k\omega_0\right)$$

(see (9.34)) then it follows

$$X\left(j\omega\right) = 2\pi \sum_{k=-\infty}^{\infty} \tilde{c}_k \delta\left(\omega - k\omega_0\right). \tag{9.38}$$

Thus, the Fourier transform of a periodic signal can be considered as a train of unit impulses occurring at the frequencies $k\omega_0$ with the weighted coefficients equal to the Fourier coefficients multiplied by 2π.

Example 9.7

Let us consider the square wave signal shown in Fig.9.25

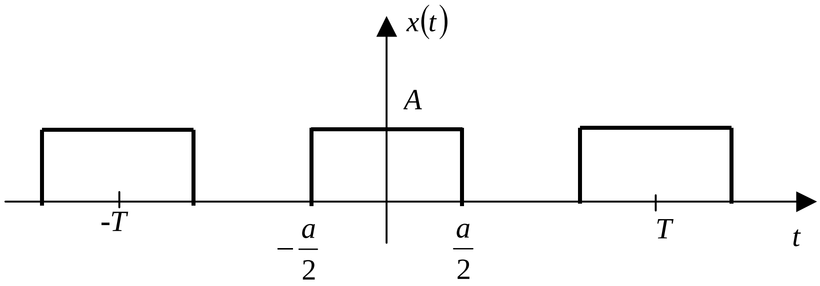

Fig. 9.25. Square wave signal for Example 9.7

The exponential Fourier series coefficients of this signal are

$$\tilde{c}_k = A\frac{a}{T}\frac{\sin k\omega_0 \frac{a}{2}}{k\omega_0 \frac{a}{2}}.$$

Hence, using (9.38) we find

$$X(\mathrm{j}\omega) = 2\pi A \frac{a}{T} \sum_{k=-\infty}^{\infty} \frac{\sin k\omega_0 \frac{a}{2}}{k\omega_0 \frac{a}{2}} \delta(\omega - k\omega_0).$$

Example 9.8

Let us consider a periodic sequence of unit impulses (see Fig.9.26).

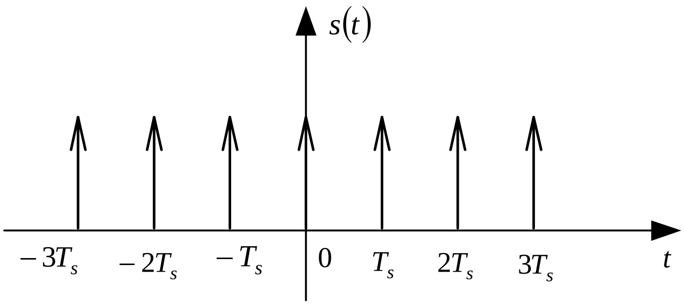

Fig. 9.26. A periodic sequence of unit impulses

This sequence has the representation

$$s(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s). \tag{9.39}$$

Let us compute the exponential Fourier series coefficients $\tilde{c}_k$

$$\tilde{c}_k = \frac{1}{T_s} \int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} \delta(t) \mathrm{e}^{-\mathrm{j}k\omega_s t} \mathrm{d}t = \frac{1}{T_s} \qquad \omega_s = \frac{2\pi}{T_s}.$$

Thus, the exponential Fourier series of $s(t)$ is

$$s(t) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} \mathrm{e}^{\mathrm{j}k\omega_s t}. \tag{9.40}$$

To find the Fourier transform of $s(t)$, we apply (9.38)

$$S(\mathrm{j}\omega)=\frac{2\pi}{T_s}\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_s)=\omega_s\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_s). \qquad (9.41)$$

The spectrum of signal $s(t)$ is a sequence of the unit impulses shifted by ω_s one from another with the strength of each impulse equal ω_s (see Fig.9.27)

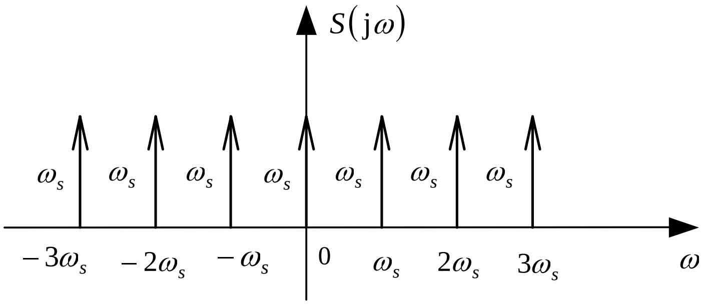

Fig. 9.27. Spectrum of the signal shown in Figure 9.26

9.7. System response in terms of Fourier transform

We will study the response of an LTI system. The system input will be denoted by $x(t)$ and the output by $y(t)$. The Fourier transforms will be denoted by $X(\mathrm{j}\omega)$ and $Y(\mathrm{j}\omega)$, respectively (see Fig.9.28).

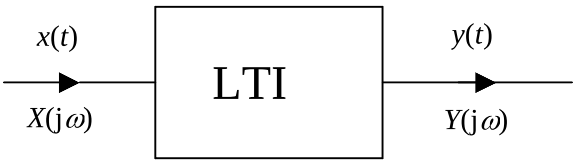

Fig. 9.28. An LTI system

The output signal $y(t)$ is related to the input signal $x(t)$ via convolution

$$y(t)=x(t)*h(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)\,\mathrm{d}\tau=\int_{-\infty}^{\infty}h(\tau)x(t-\tau)\,\mathrm{d}\tau \qquad (9.42)$$

where $h(t)$ is the impulse response of the system. The convolution theorem (9.27) gives

$$Y(j\omega) = H(j\omega)X(j\omega) \tag{9.43}$$

where $H(\mathrm{j}\omega) = \mathscr{F}(h(t))$. Thus, we have

$$H(\mathrm{j}\omega) = \int_{-\infty}^{\infty} h(t)\,\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t = \int_{0}^{\infty} h(t)\,\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t = \left.\int_{0}^{\infty} h(t)\,\mathrm{e}^{-st}\mathrm{d}t\right|_{s=\mathrm{j}\omega} = H(s)\big|_{s=j\omega}.$$

In the above rearrangements we have changed the lower limit of integration from $-\infty$ to 0 because the impulse response is zero for $t < 0$. Thus, $H(\mathrm{j}\omega)$ equals the transfer function at $s = \mathrm{j}\omega$, i. e. it is the frequency response function

$$H(j\omega) = H(s)\big|_{s=j\omega}. \tag{9.44}$$

Example 9.9

In the circuit of Fig.9.29 we determine the spectrum of the output voltage $v_0(t)$ due to the input voltage $v_i(t) = \mathrm{e}^{-2t}u(t)$.

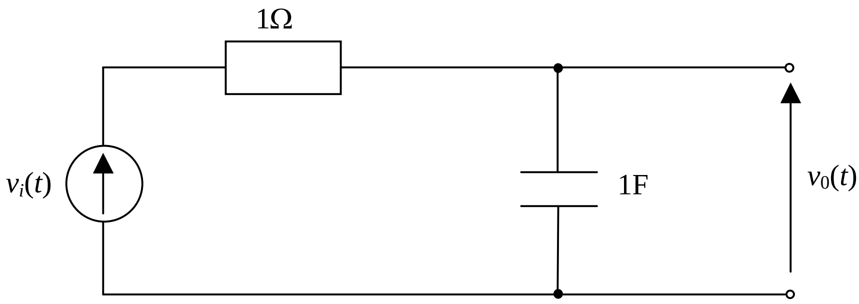

Fig. 9.29. Circuit for Example 9.9

At first, we determine the transfer function

$$H(s) = \frac{V_0(s)}{V_i(s)}.$$

Since

$$V_0(s) = \frac{V_i(s)}{1+\frac{1}{s}} \cdot \frac{1}{s} = \frac{V_i(s)}{s+1}$$

we have

$$H(s) = \frac{1}{s+1}\,.$$

Hence, we may write

$$H(\mathrm{j}\omega) = H(s)\big|_{s=\mathrm{j}\omega} = \frac{1}{1+\mathrm{j}\omega} = \frac{1}{\sqrt{1+\omega^2}}\,\mathrm{e}^{-\mathrm{j}\tan^{-1}\omega}\,. \qquad (9.45)$$

The Fourier transform of the input signal is

$$V_i(\mathrm{j}\omega) = \int_{-\infty}^{\infty} v_i(t)\,\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t = \int_{-\infty}^{\infty} \mathrm{e}^{-2t}u(t)\,\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t = \int_{0}^{\infty} \mathrm{e}^{-(2+\mathrm{j}\omega)t}\mathrm{d}t =$$

$$= -\frac{1}{2+\mathrm{j}\omega}\,\mathrm{e}^{-(2+j\omega)t}\Big|_0^{\infty} = \frac{1}{2+\mathrm{j}\omega} = \frac{1}{\sqrt{4+\omega^2}}\,\mathrm{e}^{-\mathrm{j}\tan^{-1}\frac{\omega}{2}}\,.$$

Thus the output spectrum is

$$V_0(\mathrm{j}\omega) = \left|V_0(\mathrm{j}\omega)\right|\mathrm{e}^{\mathrm{j}\phi(\omega)}$$

where:

$$\left|V_0(\mathrm{j}\omega)\right| = \frac{1}{\sqrt{(1+\omega^2)(4+\omega^2)}}$$

$$\phi(\omega) = -tan^{-1}\,\omega - tan^{-1}\,\frac{\omega}{2}\,.$$

The amplitude and phase spectra are sketched in Figs. 9.30 and 9.31.

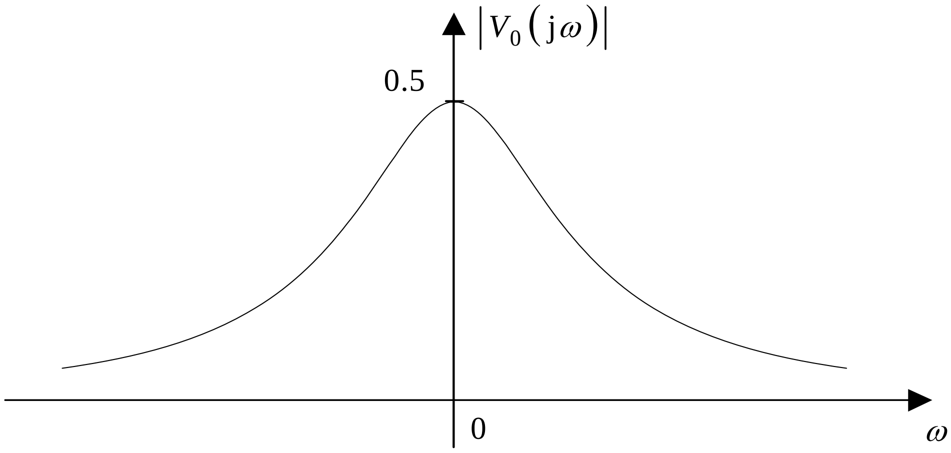

Fig. 9.30. Amplitude spectrum of $v_0(t)$

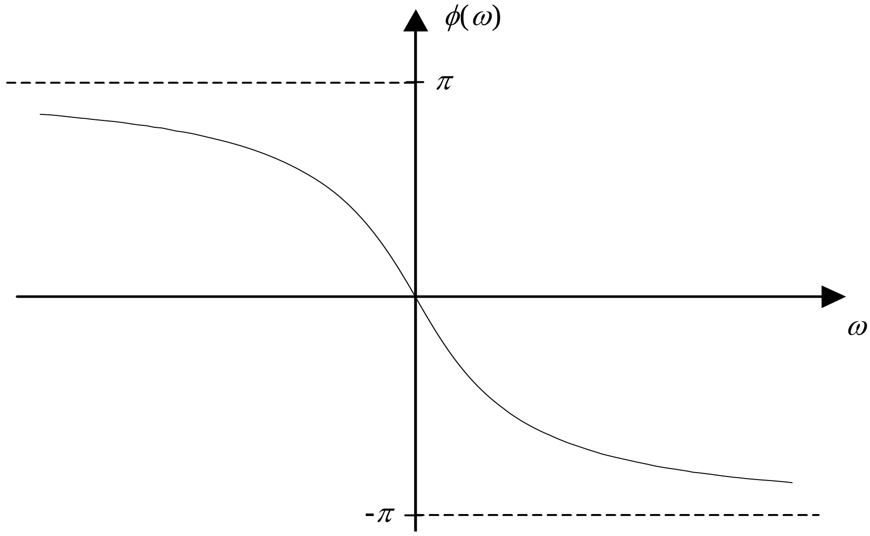

Fig. 9.31. Phase spectrum of $v_0(t)$

Example 9.10

Let us consider an LTI system specified by the frequency response function $H(j\omega) = |H(j\omega)| e^{j\phi_H(j\omega)}$ with the input signal

$$x(t) = \sum_{k=1}^{N} A_k \cos(\omega_k t + \alpha_k). \qquad (9.46)$$

To determine the output signal $y(t)$, we use (9.42). First, we find the Fourier transform of the signal $x_k(t) = A_k\cos(\omega_k t + \alpha_k)$. We rewrite this signal in the form

$$x_k(t) = A_k(\cos\omega_k t\cos\alpha_k - \sin\omega_k t\sin\alpha_k)$$

and apply (9.34)-(9.35)

$$X_k(\mathrm{j}\omega) = A_k\pi[\cos\alpha_k(\delta(\omega-\omega_k)+\delta(\omega+\omega_k))+ \\ + \mathrm{j}\sin\alpha_k(\delta(\omega-\omega_k)-\delta(\omega+\omega_k))].$$

Thus, we can determine the response of the system due to $x_k(t)$ as follows

$$Y_k(\mathrm{j}\omega) = A_k\pi|H(\mathrm{j}\omega)|\mathrm{e}^{\mathrm{j}\phi_H(\mathrm{j}\omega)}[\cos\alpha_k(\delta(\omega-\omega_k)+\delta(\omega+\omega_k))+ \\ + \mathrm{j}\sin\alpha_k(\delta(\omega-\omega_k)-\delta(\omega+\omega_k))].$$

To find $y_k(t)$, we apply the inverse Fourier transform

$$y_k(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\mathrm{j}\omega)\mathrm{e}^{\mathrm{j}\omega t}\mathrm{d}t = \frac{A_k}{2}\Big[(\cos\alpha_k + \mathrm{j}\sin\alpha_k)|H(\mathrm{j}\omega_k)|\mathrm{e}^{\mathrm{j}(\omega_k t+\phi_H(\mathrm{j}\omega_k))} + \\ + (\cos\alpha_k - \mathrm{j}\sin\alpha_k)|H(-\mathrm{j}\omega_k)|\mathrm{e}^{-\mathrm{j}(\omega_k t-\phi_H(-\mathrm{j}\omega_k))}\Big].$$

Since $|H(-\mathrm{j}\omega_k)| = |H(\mathrm{j}\omega_k)|$ and $\phi_H(-\mathrm{j}\omega_k) = -\phi_H(-\mathrm{j}\omega_k)$, then

$$\begin{aligned} y_k(t) &= \frac{A_k}{2}\Big[(\cos\alpha_k + \mathrm{j}\sin\alpha_k)|H(\mathrm{j}\omega_k)|\mathrm{e}^{\mathrm{j}(\omega_k t+\phi_H(\mathrm{j}\omega_k))} + \\ &\quad + (\cos\alpha_k - \mathrm{j}\sin\alpha_k)|H(\mathrm{j}\omega_k)|\mathrm{e}^{-\mathrm{j}(\omega_k t+\phi_H(\mathrm{j}\omega_k))}\Big] = \\ &= \frac{A_k}{2}2\mathrm{Re}\Big[(\cos\alpha_k + \mathrm{j}\sin\alpha_k)|H(\mathrm{j}\omega_k)|\mathrm{e}^{\mathrm{j}(\omega_k t+\phi_H(\mathrm{j}\omega_k))}\Big] = \\ &= A_k|H(\mathrm{j}\omega_k)|(\cos\alpha_k\cos(\omega_k t+\phi_H(\mathrm{j}\omega_k)) - \sin\alpha_k\sin(\omega_k t+\phi_H(\mathrm{j}\omega_k))) = \\ &= A_k|H(\mathrm{j}\omega_k)|\cos(\omega_k t+\alpha_k+\phi(\mathrm{j}\omega_k)). \end{aligned}$$

Hence, we obtain

$$y(t) = \sum_{k=1}^{N} A_k|H(j\omega_k)|cos(\omega_k t+\alpha_k+\phi_H(j\omega_k)). \qquad (9.46)$$