1. Basic properties of signals and systems

Question 1.1

Using the signal \( x(t) \) shown in Fig.1.1 plot the following signals:
\( 2x(-t), -x(t-3), 0.5x(2t), x(t+1), 2x(1-t). \)

![Fig.1.1](image)

Solution

Signal \( 2x(-t) \) can be obtained by folding the signal \( x(t) \) and multiplying by two (see Fig.1.2).

![Fig.1.2](image)

Signal \( -x(t-3) \) arises by deleting the signal \( x(t) \) by \( t_0 = 3 \) and reflecting in the time axis (see Fig.1.3).
To obtain signal $0.5x(2t)$ we compress $x(t)$ in time twice and multiply by 0.5 (see Fig.1.4).

Signal $x(t+1)$ proceeds signal $x(t)$ by $t_0 = 1$, as it is illustrated in Fig.1.5.
Since \( 2x(1-t) = 2x(-t - 1) \), then in order to obtain \( 2x(1-t) \) we reflect signal \( x(t) \), next delay it by \( t_0 = 1 \) and increase twice (see Fig. 1.6).

\[
2x(t) = \begin{cases} 
1 & \text{if } 0 \leq t < 1 \\
2 & \text{if } 1 \leq t < 2 \\
0 & \text{otherwise}
\end{cases}
\]

**Question 1.2**

Figure 1.7 shows a discrete signal \( x(n) \). Plot the following signals: \( 2x(-n) \), \( x(n - 1) \), \( x(1 - n) \).

**Solution**

Signal \( 2x(-n) \) can be formed by reflecting \( x(n) \) and multiplying by 2 (see Fig. 1.8).
To create signal $x(n-1)$ we delay $x(n)$ by $n_0 = 1$ as it is illustrated in Fig.1.9.

Since $x(1-n) = x(-(n-1))$, signal $x(1-n)$ arises by reflecting $x(n)$ and shifting by one to the right (see Fig.1.10).
Figures 1.11(a) and 1.11(b) show two continuous–time signals $x_1(t)$ and $x_2(t)$. Plot the following signals:

a) $y_1(t) = 2x_1(t-1) + x_2(2-t),$

b) $y_2(t) = x_1(2t - 1) + x_2(t).$
Solution

a) Signals $2x_1(t-1)$ and $x_2(2-t)$ are depicted in Fig.1.12.
Fig.1.13 shows the total signal \( y_1(t) = 2x_1(t-1) + x_2(2-t) \), obtained by adding, for all \( t \), the corresponding values of \( 2x_1(t-1) \) and \( x_2(2-t) \). Since the signals are piecewise-linear the procedure can be reduced to the breakpoints at \( t = 0, 1, 2 \).

b) To obtain the signal \( x_1(2t-1) \) we create firstly an auxiliary signal \( x_1(2t) \) (see Fig.1.14), and next write \( x_1(2t-1) = x_1(2(t-0.5)) \).

Thus, the signal \( x_1(2t-1) \) can be considered as the signal \( x_1(2t) \) shifted by 0.5 (see Fig.1.15).
The total signal $x_1(2t-1) + x_2(t)$ is shown in Fig. 1.16.

**Question 1.4**

Figure 1.17 shows two discrete signals $x_1(n)$ and $x_2(n)$. Plot the signal $y(n) = x_1(n - 1) - x_2(2 - n)$.
Solution

Signals $x_1(n-1)$ and $-x_2(2-n) = -x_2(-(n-2))$ are shown in Figs.1.18 and 1.19, respectively.

![Fig.1.18](image1)

![Fig.1.19](image2)

The total signal $y(n) = x_1(n-1) - x_2(2-n)$ is found by adding signals of figures 1.18 and 1.19 at the points $n = 0, 1, 2, 3$ (see Fig.1.20).

![Fig.1.20](image3)
Question 1.5

Decompose the signal shown in Fig.1.21 into odd and even components.

![Signal x(t)](image)

Fig.1.21. Signal $x(t)$

**Solution**

The odd term is given by $x_o(t) = 0.5[x(t) - x(-t)]$. Figures 1.22, 1.23, 1.24 show signals $0.5x(t)$, $0.5x(-t)$, $x_0(t)$, respectively. Signal $x_n(t)$ is obtained by subtracting, for all $t$, signals of figures 1.22 and 1.23.

![Signal 0.5x(t)](image)

Fig.1.22. Signal $0.5x(t)$
The even component of $x(t)$ (specified by $x_e(t) = 0.5[x(t) + x(-t)]$), found graphically by adding the signals of figures 1.22 and 1.23, is depicted in Fig.1.25.
**Question 1.6**

Decompose the discrete signal shown in Fig.1.26 into odd and even components. Plot these components versus \( n \).

![Fig.1.26](image)

**Solution**

The odd term is given by \( x_o(n) = 0.5[x(n) - x(-n)] \). Figures 1.27, 1.28, 1.29 show signals 0.5\( x(n) \), 0.5\( x(-n) \), \( x_o(n) \), respectively. Signal \( x_o(n) \) arises by subtraction signal of Fig.1.28 from the signal of Fig.1.27.

The even component of the signal \( x(n) \) has the form \( x_e(n) = 0.5[x(n) + x(-n)] \). Hence, it can be obtained by adding signals of Figs. 1.27 and 1.28 (see Fig.1.30).

![Fig.1.27](image)
Fig. 1.28

\[ 0.5x(-n) \]

\[ x_o(n) \]

Fig. 1.29
**Question 1.7**

Evaluate the following integrals:

a) \( \int_{-\infty}^{\infty} t^3 \delta(t-1) dt \),

b) \( \int_{-\infty}^{\infty} \delta(t-1) \sin 2t \, dt \),

c) \( \int_{-\infty}^{\infty} \delta(t) \cos 2t \, dt \),

d) \( \int_{-\infty}^{\infty} e^{-2t} \delta(t-1) \, dt \).

**Solution**

Using properties of the unit impulse we find:

a) \( \int_{-\infty}^{\infty} t^3 \delta(t-1) \, dt = t^3 \bigg|_{t=1} = 1 \),

b) \( \int_{-\infty}^{\infty} \delta(t-1) \sin 2t \, dt = \sin 2t \bigg|_{t=1} = \sin 2 \),

c) \( \int_{-\infty}^{\infty} \delta(t) \cos 2t \, dt = \cos 2t \bigg|_{t=0} = 1 \),

d) \( \int_{-\infty}^{\infty} e^{-2t} \delta(t-1) \, dt = e^{-2t} \bigg|_{t=1} = e^{-2} \).
**Question 1.8**

Check if the systems described by the following relations:

a) \( y(t) = x(t) + 3x(t - 1) \),

b) \( y(t) = \int_{-1}^{t} x(\tau) \, d\tau \),

c) \( y(t) = 2x(t)\sin \omega t \),

are linear or nonlinear, time invariant or time varying, instantaneous or non-instantaneous.

**Solution**

a) The response of the system to the input \( x_1(t) \) is

\[
y_1(t) = x_1(t) + 3x_1(t - 1),
\]

whereas the response due to \( x_2(t) \) is

\[
y_2(t) = x_2(t) + 3x_2(t - 1).
\]

Consider the response of the system to the input of the form

\[
x(t) = ax_1(t) + bx_2(t).
\]

Since

\[
y(t) = ax_1(t) + bx_2(t) + 3ax_1(t - 1) + 3bx_2(t - 1) = a(x_1(t) + 3x_1(t - 1)) + b(x_2(t) + 3x_2(t - 1)) = ay_1(t) + by_2(t).
\]

the system is linear.

The response of the system to

\[
x(t - T)
\]

is

\[
x(t - T) + 3x(t - 1 - T) = y(t - T).
\]

Equation (1.6) is valid for all \( t \) and arbitrary \( T \), hence, the system is time-invariant.

Since the response of the system depends on the previous input at \( t-1 \), the system is non-instantaneous.

b) We find the response of the system due to \( x_1(t) \) and \( x_2(t) \):

\[
y_1(t) = \int_{-1}^{t} x_1(\tau) \, d\tau ,
\]

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\[ y_2(t) = \int_{-1}^{t} x_2(\tau) d\tau , \]  

and the response due to the input:

\[ x(t) = ax_1(t) + bx_2(t) , \]  

\[ y(t) = \int_{-1}^{t} (ax_1(\tau) + bx_2(\tau)) d\tau = \]

\[ = a \int_{-1}^{t} x_1(\tau) d\tau + b \int_{-1}^{t} x_2(\tau) d\tau = ay_1(t) + by_2(t) . \]  

Relationship (1.10) states that the system is linear.

Next we consider signal \( x(t-T) \) and determine the response of the system to this signal

\[ \int_{-1}^{t} x(\tau - T) d\tau . \]  

Since generally

\[ y(t - T) = \int_{-1}^{t} x(\tau) d\tau \neq \int_{-1}^{t} x(\tau - T) d\tau , \]  

the system is time-varying.

The system is non-instantaneous because its response at instant \( t \) depends on all time between \(-1 \) and \( t \).

c) The responses of the system due to the inputs \( x_1(t) \) and \( x_2(t) \) are

\[ y_1(t) = 2x_1(t)\sin \omega t , \]  

\[ y_2(t) = 2x_2(t)\sin \omega t , \]  

respectively. We determine the response of the system to the input

\[ x(t) = ax_1(t) + bx_2(t) . \]  

Since, it holds

\[ y(t) = 2(ax_1(t) + bx_2(t))\sin \omega t = a2x_1(t)\sin \omega t + b2x_2(t)\sin \omega t = \]

\[ = ay_1(t) + by_2(t) . \]  

the system is linear.

The response of the system to input \( x(t-T) \), where \( T \) is an arbitrary constant, is

\[ 2x(t - T)\sin \omega t , \]  

whereas
\[ y(t - T) = 2x((t - T)\sin(\omega(t - T))). \]  

(1.17)

Since generally \(2x(t - T)\sin\omega \neq y(t - T)\), the system is time-varying. The response in the system at any instant of time depends on the input at that instant only, hence, the system is instantaneous.

**Question 1.9**

Check if the discrete systems specified by the following relationships:

a) \( y(n) = x(n) + 2(x(n))^3 \),

b) \( y(n) = x(n) + x(n-1) \),

are linear or nonlinear, time-invariant or time-varying, instantaneous or non-instantaneous.

**Solution**

a) We find the responses of the system due to the signals \(x_1(n)\) and \(x_2(n)\):

\[ y_1(n) = x_1(n) + 2(x_1(n))^3, \]  

(1.18)

\[ y_2(n) = x_2(n) + 2(x_2(n))^3, \]  

(1.19)

and due to the input \( x(n) = ax_1(n) + bx_2(n) \):

\[ y(n) = ax_1(n) + bx_2(n) + 2(ax_1(n) + bx_2(n))^3. \]  

(1.20)

Since

\[ ay_1(n) + by_2(n) = ax_1(n) + bx_2(n) + 2a(x_1(n))^3 + 2b(x_2(n))^3 \neq y(n), \]  

(1.21)

the system is nonlinear.

System response due to \(x(n-N)\) is

\[ x(n - N) + 2(x(n - N))^3, \]  

(1.22)

whereas \(y(n-N)\) is

\[ y(n - N) = x(n - N) + 2(x(n - N))^3. \]  

(1.23)

Both the relationships are identical, consequently, the system is time-invariant.

Since the output of the system at any \(n\) depends on the system input at \(n\) only the system is instantaneous.

b) The responses of the system due to the inputs \(x_1(n)\) and \(x_2(n)\) are:

\[ y_1(n) = x_1(n) + x_1(n-1), \]  

(1.24)
\[ y_2(n) = x_2(n) + x_2(n-1). \]  

(1.25)

Since the response due to input \( x(n) = ax_1(n) + bx_2(n) \) is

\[ y(n) = ax_1(n) + bx_2(n) + ax_1(n-1) + bx_2(n-1) = ay_1(n) + by_2(n), \]  

the system jest linear.

Let us consider a shifted signal \( x(n-N) \) and find the response of the system to this signal. Since the response fulfils the equality

\[ x(n-N) + x(n-1-N) = y(n-N), \]  

(1.27)

the system is time-invariant.

The response of the system at arbitrary \( n \) depends on the input both at \( n \) and on \( n-1 \). Therefore, the system is non-instantaneous.

**Question 1.10**

Let us consider a LTI system with the input \( x(t) \) and output \( y(t) \) as shown in Fig.1.30.

![Fig.1.31](image)

Find the responses of this system due to the inputs \( x_1(t) \) and \( x_2(t) \) shown in Fig.1.32.

![Fig.1.32](image)

**Solution**

The analytical forms of \( x_1(t) \) and \( x_2(t) \) are:
\[ x_1(t) = 0.5x(t+1) + x(t) - 0.5x(t-1), \]
\[ x_2(t) = x(t) - x(t-1). \]

Figure 1.33 shows the responses of the system due to any of the three parts of \( x_1(t) \) having the form of rectangular pulses. Since the system is LTI we add together these responses
\[ y_1(t) = y_1^{(1)}(t) + y_1^{(2)}(t) + y_1^{(3)}(t). \]

The total response \( y_1(t) \) is shown in Fig.1.34(a).

Similarly we find \( y_2(t) \) (see Fig.1.34b).

**Question 1.11**

Using graphical interpretation of the convolution find the zero–state response at \( t = 0.5 \) of a linear time invariant system specified by the impulse response
\[ h(t) = \begin{cases} 
3e^{-t}, & 0 \leq t < 1, \\
0, & t \geq 1, 
\end{cases} \quad (1.28) \]
due to the input
\[ e(t) = \begin{cases} 
2, & 0 \leq t < 0.5, \\
0, & t \geq 0.5.
\end{cases} \quad (1.29) \]

**Solution**

The zero–state response is given by the convolution of \( e(t) \) and \( h(t) \)

\[ r(t) = \int_{0}^{\infty} e(\tau)h(t - \tau)d\tau. \quad (1.30) \]

Figures 1.35c, d, e show three steps of the convolution operation: folding, translating, and multiplying.

![Convolution Diagrams](image)

The last step is the integration as follows
\[
\int_0^{0.5} h(0.5 - \tau) \cdot e(\tau) \, d\tau = \int_0^{0.5} 3e^{-0.5(0.5 - \tau)} \, d\tau = 6e^{-0.5} \int_0^{0.5} e^\tau \, d\tau = 6 - 6e^{-0.5} = 2.36. \tag{1.31}
\]

**Question 1.12**

Calculate and sketch the function that results from convolving the pair of functions shown in Fig.1.36.

![f_1(t) and f_2(t)](image)

**Solution**

The convolution of \(f_1(t)\) and \(f_2(t)\) is given by

\[
g(t) = \int_0^\infty f_1(\tau)f_2(t - \tau) \, d\tau, \quad t \geq 0. \tag{1.32}
\]

Fig.1.37 shows the graphical construction enabling us to find the convolution at \(t = 0.5\). The convolution at \(t = 0.5\) is the area under the plot \(f_1(\tau)f_2(0.5 - \tau)\) (see Fig.1.37d). This area is equal to 1.125, hence, we have \(g(0.5) = 1.125\). Repeating this procedure for \(t = 1.0, 1.5, 2.0, 2.5, 3.0\) we obtain the results summarised in Table 1.

<table>
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<tr>
<th>(t)</th>
<th>0.5</th>
<th>1.0</th>
<th>1.5</th>
<th>2.0</th>
<th>2.5</th>
<th>3.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>(g(t))</td>
<td>1.125</td>
<td>1.5</td>
<td>-0.75</td>
<td>-1.5</td>
<td>-0.375</td>
<td>0</td>
</tr>
</tbody>
</table>
Fig. 1.37

(a) $f_1(\tau)$

(b) $f_2(-\tau)$

(c) $f_2(0.5-\tau)$

(d) $f_1(\tau)f_2(0.5-\tau)$
On the basis of these results we plot $g(t)$ as depicted in Fig.1.38.

![Fig.1.38](image)

**Question 1.13**

Repeat **Question 1.12** using analytical approach.

**Solution**

Analytical description of $f_1(t)$ is

$$f_1(t) = \begin{cases} 1 - t , & t \in [0;1], \\ 0 , & t \notin [0;1]. \end{cases} \quad (1.33)$$

We calculate the convolution in four time intervals as follows:

$$g(t) = \int_0^t (1 - \tau)3d\tau = 3t - \frac{3}{2}t^2 , \quad t \in [0;1], \quad (1.34)$$

$$g(t) = \int_0^{t-1} (1 - \tau)(-3)d\tau + \int_{t-1}^{1} (1 - \tau)3d\tau = 3t^2 - 12t + 10.5 , \quad t \in [1;2], \quad (1.35)$$

$$g(t) = \int_{t-2}^{1} (1 - \tau)(-3)d\tau = -\frac{3}{2}t^2 + 9t - 13.5 , \quad t \in [2;3], \quad (1.36)$$

$$g(t) = 0 . \quad t \in [3;\infty] , \quad (1.37)$$
**Question 1.14**

Calculate and sketch the convolution of a pair of functions shown in Fig.1.39 using the graphical approach.

![Fig.1.39](image)

**Solution**

We use the convolution formula

\[ g(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t - \tau) \, d\tau. \]  

(1.38)

![Fig.1.40](image)

Fig.1.40 shows three steps of the convolution operation, namely folding, translating, and multiplying at \( t = 0.2 \). Hence, we have \( g(0.2) = 0.04 \). Repeating the procedure for other times we
obtain the results summarised in Table 2. Plot of \( g(t) \) is depicted in Fig.1.41.

<table>
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<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1.0</th>
<th>1.2</th>
<th>1.4</th>
<th>1.6</th>
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<tr>
<td>( g(t) )</td>
<td>0.04</td>
<td>0.16</td>
<td>0.36</td>
<td>0.64</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
2. The Laplace transform

Question 2.1

Find the Laplace transform of the following functions:

a) \( f_1(t) = 5 + 3e^{-2t} \),

b) \( f_2(t) = e^{-3t} + \cos 5t \),

c) \( f_3(t) = 3 + 2t + 4t^2 \),

d) \( f_4(t) = 7\sin 2t + \cos 3t \),

e) \( f_5(t) = 2\sin 10t + 30^\circ \).

Solution

a) The Laplace transform of the function \( f_1(t) \) is given by

\[
F_1(s) = \int_0^\infty f_1(t)e^{-st}dt = \int_0^\infty 5e^{-st}dt + \int_0^\infty 3e^{-2t}e^{-st}dt .
\]

The first integral on the right hand side is

\[
I_1 = \int_0^\infty 5e^{-st}dt = -5 \frac{e^{-st}}{s} \bigg|_0^\infty = -5 e^{-\sigma} e^{-j\omega} \bigg|_0^\infty ,
\]

where we substitute \( s = \sigma + j\omega \). For any positive \( \sigma \) it holds

\[
\lim_{t \to \infty} \left( -5 \frac{e^{-\sigma} e^{-j\omega}}{s} \right) = 0 .
\]

Hence, we obtain

\[
I_1 = \frac{5}{s} .
\]

The other integral is

\[
I_2 = \int_0^\infty 3e^{-(s+2)t}dt = -\frac{3}{s+2} e^{-(s+2)t} \bigg|_0^\infty = -\frac{3}{s+2} e^{-(\sigma+2)\omega} e^{-j\omega} \bigg|_0^\infty .
\]

For \( \sigma + 2 > 0 \), it holds

\[
\lim_{t \to \infty} e^{-(\sigma+2)\omega} = 0 .
\]

Hence, we have

\[
I_2 = \frac{3}{s+2} .
\]

Substituting (2.2) and (2.3) into (2.1) we obtain

\[
F_1(s) = \frac{5}{s} + \frac{3}{s+2} ,
\]
where the region of convergence is $\sigma = \text{Re}(s) > 0$.

b) The Laplace transform of the first term of $f_2(t)$ is

$$\mathcal{L}(e^{-3t}) = \frac{1}{s + 3}. \quad (2.5)$$

To find the Laplace transform of the other term we express $\cos 5t$ in terms of the exponential functions

$$\cos 5t = \frac{e^{5jt} + e^{-5jt}}{2}$$

and apply the linearity property

$$\mathcal{L}(\cos 5t) = \frac{1}{2} \mathcal{L}(e^{5jt}) + \frac{1}{2} \mathcal{L}(e^{-5jt}) = \frac{1}{2} \frac{1}{s - j5} + \frac{1}{2} \frac{1}{s + j5} = \frac{s}{s^2 + 25}. \quad (2.6)$$

Combining (2.5) and (2.6) gives

$$F_2(s) = \frac{1}{s + 3} + \frac{s}{s^2 + 25}.$$

c) The Laplace transform of the first term is

$$\mathcal{L}(3) = \frac{3}{s}. \quad (2.7)$$

To determine the Laplace transform of the second term we consider the unit step function and compute the integral

$$r(t) = \int_0^t u(\tau) \, d\tau = tu(t). \quad (2.8)$$

Next, we use the integration rule

$$\mathcal{L}(r(t)) = \frac{1}{s} \mathcal{L}(u(t)) = \frac{1}{s^2}. \quad (2.9)$$

Hence, we have

$$\mathcal{L}(2t) = \mathcal{L}(2r(t)) = \frac{2}{s^2}. \quad (2.10)$$

To find the Laplace transform of the third term we compute the integral of the function $r(t)$

$$r_1(t) = \int_0^t r(\tau) \, d\tau = \int_0^t u(\tau) \, d\tau = \frac{t^2}{2} u(t) \quad (2.11)$$

and apply the integration rule

$$\mathcal{L}\left(\frac{t^2}{2} u(t)\right) = \mathcal{L}(r_1(t)) = \frac{1}{s} \mathcal{L}(r(t)) = \frac{1}{s^3}. \quad (2.12)$$

Hence, it follows

$$\mathcal{L}(4t^2) = \frac{8}{s^3}. \quad (2.12)$$

On the basis of (2.7), (2.10) and (2.12) we obtain
To find the Laplace transform of the function \( \sin 2t \) we use the identity

\[
\sin 2t = \frac{e^{j2t} - e^{-j2t}}{2j}
\]

and the formula

\[
\mathcal{L}(e^{at}) = \frac{1}{s - a}.
\]

Hence, it follows

\[
\mathcal{L}(7\sin 2t) = \frac{7}{2j} \frac{1}{s - 2j} - \frac{7}{2j} \frac{1}{s + 2j} = \frac{14}{s^2 + 4}.
\]

Similarly, we find

\[
\mathcal{L}(\cos 3t) = \frac{s}{s^2 + 9}.
\]

Using (2.13) and (2.14) we obtain

\[
F_4(s) = \frac{14}{s^2 + 4} + \frac{s}{s^2 + 9}.
\]

e) Function \( f_5(t) \) can be expressed in terms of the exponential functions as follows:

\[
f_5(t) = 2 \frac{e^{j(\pi r + 30^\circ)} - e^{-j(\pi r + 30^\circ)}}{2j} = \frac{1}{j} \left( e^{j30^\circ} e^{j\pi r} - e^{-j30^\circ} e^{-j\pi r} \right).
\]

Applying the Laplace transform to the above function yields

\[
F_5(s) = \frac{1}{j} \frac{e^{j30^\circ}}{s - j\pi 0} - \frac{1}{j} \frac{e^{-j30^\circ}}{s + j\pi 0} = \frac{1}{j} \frac{e^{j30^\circ}}{s - j\pi 0} - \frac{1}{j} \frac{e^{-j30^\circ}}{s + j\pi 0} = \frac{1}{j} \left( \cos 30^\circ + js30^\circ \right) \frac{(s + j\pi 0) - (s - j\pi 0)}{s^2 + 100} = \frac{1}{j} \left( \frac{\sqrt{3}}{2} + j \frac{1}{2} \right) \frac{(s + j\pi 0) - (s - j\pi 0)}{s^2 + 100} = \frac{1}{j} \frac{s + 10\sqrt{3}}{s^2 + 100}.
\]

Alternatively, \( F_5(s) \) can be found as follows. We rearrange \( f_5(t) \) using the trigonometric identity

\[
\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta.
\]

As a result we obtain
Applying the Laplace transform yields

\[ F_3(s) = \frac{\sqrt{3}}{s^2 + 100} + \frac{s}{s^2 + 100} = \frac{s + 10\sqrt{3}}{s^2 + 100}. \]

**Question 2.2**

Given that \( \mathcal{L}(f(t)) = F(s) \) and that \( b \) is a positive constant prove that

\begin{align*}
\text{a)} & \quad \mathcal{L}(e^{-bt} f(t)) = F(s + b), & (2.15) \\
\text{b)} & \quad \mathcal{L}(f(bt)) = \frac{1}{b} F\left(\frac{s}{b}\right). & (2.16)
\end{align*}

**Solution**

\( \text{a)} \)  We apply the Laplace transform to the function \( e^{-bt} f(t) \)

\[ \mathcal{L}(e^{-bt} f(t)) = \int_0^\infty e^{-bt} f(t) e^{-st} \, dt = \int_0^\infty f(t) e^{-(s+b)t} \, dt = F(s + b). \]

\( \text{b)} \)  First we apply the Laplace integral

\[ \mathcal{L}(f(bt)) = \int_0^\infty f(bt) e^{-st} \, dt = \int_0^\infty f(t) e^{\frac{s}{b}bt} \, dt. \]

Let \( \tilde{s} = \frac{s}{b} \) and \( \tilde{t} = bt \), then

\[ \mathcal{L}(f(t)) = \int_0^\infty f(t) e^{-\tilde{s}\tilde{t}} \, d\tilde{t} = \frac{1}{b} F(\tilde{s}) = \frac{1}{b} F\left(\frac{s}{b}\right). \]

**Question 2.3**

Find the Laplace transform of the following functions:

\begin{align*}
\text{a)} & \quad f_1(t) = 3u(t - 2)e^{-(t-2)}, \\
\text{b)} & \quad f_2(t) = 4e^{-3t}\cos5t + 2\delta(t - 1).
\end{align*}

**Solution**

\( \text{a)} \)  Function \( f_1(t) \) can be presented in the form

\[ f_1(t) = g(t - 2), \]
where
\[ g(t) = 3u(t)e^{-t} . \]

Using shifting theorem we have
\[ \mathcal{L}(f_1(t)) = e^{-2s}\mathcal{L}(g(t)) = 3e^{-2s}\frac{1}{s+1} . \]

b) To find the Laplace transform of the first term we apply the formula (2.15)
\[ \mathcal{L}(4e^{3t}\cos 5t) = 4F(s + 3) , \]
where
\[ F(s) = \mathcal{L}(\cos 5t) = \frac{s}{s^2 + 25} . \]

Hence, we have
\[ \mathcal{L}(4e^{3t}\cos 5t) = 4\frac{s + 3}{(s + 3)^2 + 25} . \quad (2.17) \]

Next we consider the other term and use the shifting theorem
\[ \mathcal{L}(2\delta(t - 1)) = 2e^{-s}\mathcal{L}(\delta(t)) = 2e^{-s} . \quad (2.18) \]
Combining (2.17) and (2.18) we obtain
\[ F_2(s) = 4\frac{s + 3}{(s + 3)^2 + 25} + 2e^{-s} . \]

**Question 2.4**

![Fig. 2.1](image-url)
Find the Laplace transforms of the functions shown in Fig. 2.1.

**Solution**

a) Function \( f_1(t) \) can be expressed in terms of \( u(t) \) as follows

\[
f_1(t) = u(t) - 2u(t - 1) + u(t - 2).
\]

Using the shifting theorem we obtain

\[
F_1(s) = \frac{1}{s} - 2e^{-s} + \frac{1}{s} e^{-2s}.
\]

b) Function \( f_2(t) \) has the following description

\[
f_2(t) = t(u(t) - u(t - 1)) = tu(t) - (t - 1)u(t - 1) - u(t - 1).
\]

Hence, we have

\[
F_2(s) = \frac{1}{s^2} - \frac{1}{s^2} e^{-s} - \frac{1}{s} e^{-s}.
\]

c) Function \( f_3(t) \) can be specified as follows

\[
f_3(t) = (-1 + u(t) - u(t - 1)) = -tu(t) + u(t) + (t - 1)u(t - 1)
\]

and its Laplace transform is

\[
F_3(s) = -\frac{1}{s^2} + \frac{1}{s} + \frac{1}{s^2} e^{-s}.
\]

d) We express the function \( f_4(t) \) in terms of \( f_2(t) \) and \( f_3(t) \)

\[
f_4(t) = f_2(t) + f_3(t - 1)u(t - 1).
\]

Using the results of part b) and c) and the shifting theorem we obtain

\[
F_4(s) = F_2(s) + e^{-s}F_3(s) = \frac{1}{s^2} - \frac{1}{s^2} e^{-s} - \frac{1}{s} e^{-s} - \frac{1}{s^2} e^{-s} + \frac{1}{s} e^{-s} + \frac{1}{s^2} e^{-2s} = \frac{1}{s^2} - \frac{2}{s} e^{-s} + \frac{1}{s^2} e^{-2s}.
\]

e) Function \( f_5(t) \) can be described as follows

\[
f_5(t) = u(t) + u(t - 1) - 2u(t - 2).
\]

Hence, using the shifting theorem we have

\[
F_5(s) = \frac{1}{s} + \frac{1}{s} e^{-s} - \frac{2}{s} e^{-2s}.
\]
Similarly as in part e) we find

\[ f_e(t) = 2u(t) + 2u(t-1) - 2u(t-2) - 2u(t-3) \]

and

\[ F_e(s) = \frac{2}{s} + \frac{2}{s}e^{-s} - \frac{2}{s}e^{-2s} - \frac{2}{s}e^{-3s}. \]

**Question 2.5**

Using the Laplace transform method solve the following differential equations:

a) \[ \frac{d^2x}{dt^2} + 2 \frac{dx}{dt} + x = 0, \quad x(0) = 1, \quad \frac{dx}{dt} \bigg|_{t=0} = 1, \]

b) \[ \frac{d^2x}{dt^2} + 3x = 0, \quad x(0) = 1, \quad \frac{dx}{dt} \bigg|_{t=0} = 0, \]

c) \[ \frac{dx}{dt} + 2x = e^{-2t}, \quad x(0) = -3. \]

**Solution**

a) We apply the Laplace transform to both sides of the equation

\[ \mathcal{L}\left(\frac{d^2x}{dt^2}\right) + 2 \mathcal{L}\left(\frac{dx}{dt}\right) + X(s) = 0 \quad (2.19) \]

and use the differentiation property:

\[ \mathcal{L}\left(\frac{dx}{dt}\right) = sX(s) - x(0) = sX(s) - 1, \quad (2.20) \]

\[ \mathcal{L}\left(\frac{d^2x}{dt^2}\right) = s^2X(s) - sx(0) - \frac{dx}{dt} \bigg|_{t=0} = s^2X(s) - s - 1. \quad (2.21) \]

Next we substitute (2.20) and (2.21) into (2.19)

\[ s^2X(s) - s - 1 + 2sX(s) - 2 + X(s) = 0. \quad (2.22) \]

Equation (2.22) is solved for \( X(s) \)

\[ X(s) = \frac{s + 3}{s^2 + 2s + 1} = \frac{s + 3}{(s + 1)^2} \]

and rearranged as follows

\[ X(s) = \frac{s + 1}{(s + 1)^2} + \frac{2}{(s + 1)^2} = \frac{1}{s + 1} + \frac{2}{(s + 1)^2}. \]
The first term on the right hand side is the Laplace transform of the function $e^{-t}$. To find the time function, having the Laplace transform equal to the second term, we use (2.15) repeated below

$$\mathcal{L}(e^{bt}f(t)) = F(s + b)$$

and

$$\mathcal{L}(t) = \frac{1}{s^2}.$$  

Hence, we have

$$\mathcal{L}(e^{-t}) = \frac{1}{(s + 1)^2}$$

and

$$x(t) = e^{-t} + 2te^{-t}, \quad t \geq 0.$$

b) Similarly as in part a) we find

$$s^2X(s) - s + 3X(s) = 0$$

and next

$$X(s) = \frac{s}{s^2 + 3} = \frac{s}{s^2 + (\sqrt{3})^2}.$$  \hspace{1cm} (2.23)

We recognise the expression on the right hand side of (2.23) as the Laplace transform of $\cos \sqrt{3}t$. Thus, the solution is

$$x(t) = \cos \sqrt{3}t, \quad t \geq 0.$$  

C) Since

$$\mathcal{L}\left(\frac{dx}{dt}\right) = sx(s) - x(0) = sx(s) + 3$$

and

$$\mathcal{L}(e^{-2t}) = \frac{1}{s + 2},$$

then

$$sX(s) + 3 + 2X(s) = \frac{1}{s + 2}.$$  

or

$$X(s) = \frac{1}{(s + 2)^2} - \frac{3}{s + 2}.$$  \hspace{1cm} (2.24)

Using the relationships

$$\mathcal{L}(f(t)e^{-kt}) = F(s + k)$$

and

$$\mathcal{L}(t) = \frac{1}{s^2},$$

we find

$$\frac{1}{(s + 2)^2} = \mathcal{L}(te^{-2t}).$$  \hspace{1cm} (2.25)
Furthermore, it holds

\[ \frac{3}{s + 2} = \mathcal{L}\{3e^{-2t}\}. \]  \hspace{1cm} (2.26)

On the basis of (2.24) – (2.26) we obtain

\[ x(t) = te^{-2t} - 3e^{-2t}, \quad t \geq 0. \]

**Question 2.6**

Using the partial fraction expansion find the inverse Laplace transform of the following functions:

a) \( F_1(s) = \frac{s^2 + 2s + 3}{s^2 + 4s + 3} \),

b) \( F_2(s) = \frac{s + 1}{(s + 2)(s + 3)} \),

c) \( F_3(s) = \frac{1}{(s^2 + 1)(s + 1)} \),

d) \( F_4(s) = \frac{s}{(s + 2)^2(s + 3)} \),

e) \( F_5(s) = \frac{1 + e^{-2s}}{s^2} \).

**Solution**

a) Since the polynomials in the numerator and the denominator have the same degree we divide \( s^2 + 2s + 3 \) by \( s^2 + 4s + 3 \)

\[ F_1(s) = 1 - \frac{2s}{s^2 + 4s + 3}. \]

Using the inverse Laplace transform we find

\[ f_1(t) = \delta(t) - 2\mathcal{L}^{-1}\left(\frac{s}{s^2 + 4s + 3}\right). \]  \hspace{1cm} (2.27)

Function

\[ F_{11}(s) = \frac{s}{s^2 + 4s + 3}. \]
has two poles $p_1 = -1$, $p_2 = -3$, hence, its partial fraction expansion is

$$F_{11}(s) = \frac{k_1}{s + 1} + \frac{k_2}{s + 3},$$  \hspace{1cm} (2.28)

where

$$k_1 = \lim_{s \to -1} \frac{s}{s + 3} = -\frac{1}{2},$$

$$k_2 = \lim_{s \to -3} \frac{s}{s + 1} = \frac{3}{2}.$$

The inverse Laplace transform of $F_{11}(s)$, given by (2.28), is

$$\mathcal{L}^{-1}(F_{11}(s)) = -\frac{1}{2}e^{-t} + \frac{3}{2}e^{-3t}.$$  \hspace{1cm} (2.29)

Substituting (2.29) to (2.27) we obtain

$$f_1(t) = \delta(t) + e^{-t} - 3e^{-3t}, \quad t \geq 0.$$

b) Partial fraction expansion of $F_2(s)$ is

$$F_2(s) = \frac{k_1}{s + 2} + \frac{k_2}{s + 3},$$

where

$$k_1 = \lim_{s \to -2} \frac{s + 1}{s + 3} = -1,$$

$$k_2 = \lim_{s \to -3} \frac{s + 1}{s + 2} = 2.$$

Hence,

$$f_2(t) = \mathcal{L}^{-1}\left(-\frac{1}{s + 2}\right) + \mathcal{L}^{-1}\left(\frac{2}{s + 3}\right) = -2e^{-2t} + 2e^{-3t}, \quad t \geq 0.$$

c) Poles of function $F_3(s)$ are $p_1 = j$, $p_2 = -j$, $p_3 = -1$ and its partial fraction expansion is

$$F_3(s) = \frac{1}{(s - j)(s + j)(s + 1)} = \frac{k_1}{s - j} + \frac{k_2}{s + j} + \frac{k_3}{s + 1},$$  \hspace{1cm} (2.30)

where

$$k_1 = \lim_{s \to j} \frac{1}{(s + j)(s + 1)} = -\frac{1}{4}(1 + j),$$

$$k_2 = k_1^* = -\frac{1}{4}(1 - j),$$

$$k_3 = \lim_{s \to -1} \frac{1}{s^2 + 1} = \frac{1}{2}.$$
We substitute the factors $k_1, k_2, k_3$ into (2.30) and find the inverse Laplace transform

$$f_3(t) = -\frac{1}{4}(1+j)\mathcal{L}^{-1}\left(\frac{1}{s-j}\right) - \frac{1}{4}(1-j)\mathcal{L}^{-1}\left(\frac{1}{s+j}\right) + \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) = -\frac{1}{4}(1+j)e^{jt} - \frac{1}{4}(1-j)e^{-jt} + \frac{1}{2}e^{-t}, \quad t \geq 0.$$ 

Note that at any fixed $t$ the first and the second terms on the right hand side form a pair of complex conjugate numbers. Hence, their sum is the double real part of the number. Using this property we have

$$f_3(t) = -\frac{1}{2}\text{Re}\left((1+j)e^{jt}\right) + \frac{1}{2}e^{-t} = -\frac{1}{2}\cos t + \frac{1}{2}\sin t + \frac{1}{2}e^{-t}.$$

d) Function $F_4(t)$ has a double pole $p_1 = -2$ and a simple pole $p_2 = -3$. The partial fraction expansion of $F_4(t)$ is

$$F_4(s) = \frac{k_{11}}{(s+2)^2} + \frac{k_{12}}{s+2} + \frac{k_2}{s+3}, \quad (2.31)$$

where

$$k_{11} = \lim_{s \to -2} (s+2)^2F_4(s) = \lim_{s \to -2} \frac{s}{s+3} = -2,$$

$$k_{12} = \lim_{s \to -2} \frac{d}{ds} (s+2)^2F_4(s) = \lim_{s \to -2} \frac{3}{(s+3)^2} = 3,$$

$$k_2 = \lim_{s \to -3} \frac{s}{s+2} = -3.$$

Hence, the inverse Laplace transform of the partial fraction expansion (2.31) is

$$f_4(t) = -2\mathcal{L}^{-1}\left(\frac{1}{(s+2)^2}\right) + 3\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) - 3\mathcal{L}^{-1}\left(\frac{1}{s+3}\right) = -2te^{-2t} + 3e^{-2t} - 3e^{-3t}, \quad t \geq 0.$$ 

e) Function $F_5(s)$ can be rewritten as follows

$$F_5(s) = \frac{1}{s^2} + \frac{1}{s^2}e^{-2s}.$$

Since $\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = tu(t)$ and $\mathcal{L}^{-1}\left(\frac{1}{s^2}e^{-2s}\right) = (t-2)u(t-2)$, we obtain

$$f_5(t) = tu(t) + (t-2)u(t-2), \quad t \geq 0.$$ 

**Question 2.7**

Using the Laplace transform method solve the following differential equations:
a) \[ \frac{d^2x}{dt^2} + \frac{dx}{dt} + 2x = u(t), \quad x(0) = 1, \quad \frac{dx}{dt}_{t=0} = 0, \]
b) \[ \frac{dx}{dt} + 2x = \cos t, \quad x(0) = 0. \]

Solution

a) We apply the Laplace transform to both sides of the equation:

\[ s^2X(s) - sX(s) - 1 + 2X(s) = \frac{1}{s}. \]

After simple rearrangement we obtain

\[ X(s) = \frac{1 + s}{s^2 + s + 2} = \frac{s^2 + s + 1}{s(s^2 + s + 2)}. \] (2.32)

To find the inverse Laplace transform of \( X(s) \) we form its partial fraction expansion

\[ \frac{s^2 + s + 1}{s(s^2 + s + 2)} = \frac{s^2 + s + 1}{s \left( s - \frac{1 + j\sqrt{7}}{2} \right) \left( s - \frac{1 - j\sqrt{7}}{2} \right)} = \]

\[ = \frac{k_1}{s} + \frac{k_2}{s - \frac{1 + j\sqrt{7}}{2}} + \frac{k_3}{s - \frac{1 - j\sqrt{7}}{2}}, \] (2.33)

where

\[ k_1 = \lim_{s \to 0} \frac{s^2 + s + 1}{s^2 + s + 2} = \frac{1}{2}, \]

\[ k_2 = \lim_{s \to \frac{1 + j\sqrt{7}}{2}} \frac{s^2 + s + 1}{s \left( s - \frac{1 + j\sqrt{7}}{2} \right)} = \frac{1}{4} - \frac{j\sqrt{7}}{28}, \]

\[ k_3 = \lim_{s \to \frac{1 - j\sqrt{7}}{2}} \frac{s^2 + s + 1}{s \left( s - \frac{1 - j\sqrt{7}}{2} \right)} = \frac{1}{4} + \frac{j\sqrt{7}}{28} = k_2^* \]

Using (2.33) and the coefficients \( k_1, k_2, k_3 \) we obtain

\[ x(t) = \frac{1}{2} \mathcal{L}^{-1} \left( \frac{1}{s} \right) + \left( \frac{1}{4} - \frac{j\sqrt{7}}{28} \right) \mathcal{L}^{-1} \left( \frac{1}{s - \frac{1 + j\sqrt{7}}{2}} \right) + \left( \frac{1}{4} + \frac{j\sqrt{7}}{28} \right) \mathcal{L}^{-1} \left( \frac{1}{s - \frac{1 - j\sqrt{7}}{2}} \right) = \]

\[ = \frac{1}{2} u(t) + \left( \frac{1}{4} - \frac{j\sqrt{7}}{28} \right) e^{-\frac{1 + j\sqrt{7}t}{2}} + \left( \frac{1}{4} + \frac{j\sqrt{7}}{28} \right) e^{-\frac{1 - j\sqrt{7}t}{2}}. \]
Note that at any fixed \( t \) the second and the third terms on the right hand side form a pair of complex conjugate numbers. Hence, their sum is equal to the double real part of the number, i.e.

\[
2\text{Re}\left(\frac{1}{4} - j\frac{\sqrt{7}}{28}\right) e^{-\frac{1}{2}t} = 2\text{Re}\left[\left(\frac{1}{4} - j\frac{\sqrt{7}}{28}\right) e^{-\frac{1}{2}t} \left(\cos\frac{\sqrt{7}}{2} t + j\sin\frac{\sqrt{7}}{2} t\right)\right] = \left(\frac{1}{2}\cos\frac{\sqrt{7}}{2} t + \frac{\sqrt{7}}{14} \sin\frac{\sqrt{7}}{2} t\right) e^{-\frac{1}{2}t}.
\]

Thus, we have

\[
x(t) = \frac{1}{2} u(t) + \left(\frac{1}{2}\cos\frac{\sqrt{7}}{2} t + \frac{\sqrt{7}}{14} \sin\frac{\sqrt{7}}{2} t\right) e^{-\frac{1}{2}t} \quad t \geq 0. \tag{2.34}
\]

b) Similarly as in part a) we apply the Laplace transform to both sides of the equation, obtaining

\[
sX(s) + 2X(s) = \frac{s}{s^2 + 1}
\]

and solve for \( X(s) \)

\[
X(s) = \frac{s}{(s^2 + 1)(s + 2)} = \frac{s}{s(s+j)(s-j)(s+2)} = \frac{k_1}{s+j} + \frac{k_2}{s-j} + \frac{k_3}{s+2}, \tag{2.35}
\]

where

\[
k_1 = \lim_{s \to -j} \frac{s}{(s-j)(s+2)} = 0.2 + j0.1,
\]

\[
k_2 = \lim_{s \to j} \frac{s}{(s+j)(s+2)} = 0.2 - j0.1 = k_1^*.
\]

\[
k_3 = \lim_{s \to -2} \frac{s}{s^2 + 1} = -0.4.
\]

We make a substitution for \( k_1, k_2, k_3 \) in (2.35) and find the inverse Laplace transform

\[
x(t) = \mathcal{L}^{-1}(X(s)) = (0.2 + j0.1)e^{-jt} + (0.2 - j0.1)e^{jt} - 0.4e^{-2t} = 2\text{Re}\left[(0.2 + j0.1)e^{-jt}\right] - 0.4e^{-2t} = 0.4\cos t + 0.2\sin t - 0.4e^{-2t} \quad t \geq 0.
\]

**Question 2.8**

Calculate the zero–state response for given transfer function \( H(s) \) and the input signal \( e(t) \)

a) \( H(s) = \frac{s + 3}{s^2 + 5s + 6}, \quad e(t) = (5\sin t)u(t), \)
b) \( H(s) = \frac{s + 1}{s^2 + 36}, \quad e(t) = (e^{-2t})u(t). \)

**Solution**

a) Let us denote the response signal by \( y(t) \). Using the definition of transfer function we write

\[
y(t) = \mathcal{L}^{-1}\{H(s)E(s)\}. \tag{2.36}
\]
Setting
\[ E(s) = \mathcal{L}\{5\sin t\} = \frac{5}{s^2 + 1} \quad (2.37) \]

and the transfer function into (2.36) yields
\[ y(t) = \mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left\{ \frac{5(s + 3)}{(s^2 + 5s + 6)(s^2 + 1)} \right\} = \mathcal{L}^{-1}\left\{ \frac{5}{(s + 2)(s + j)} \right\}. \quad (2.38) \]

The partial fraction expansion of \( Y(s) \) is
\[ Y(s) = \frac{5}{(s + 2)(s + j)} = \frac{k_1}{s + 2} + \frac{k_2}{s + j} + \frac{k_3}{s + j}, \quad (2.39) \]

where the coefficients \( k_1, k_2, k_3 \) are:
\[ k_1 = \lim_{s \to -2} \frac{5}{s^2 + 1} = 1, \quad (2.40) \]
\[ k_2 = \lim_{s \to -j} \frac{5}{(s + 2)(s + j)} = \frac{5}{2(2j - 1)} = \frac{1}{2}(1 + 2j), \quad (2.41) \]
\[ k_3 = k_2^* = -\frac{1}{2}(1 - 2j). \quad (2.42) \]

Now, we find the zero–state response
\[ y(t) = e^{-2t} + 2\Re\left\{ -\frac{1}{2}(1 + 2j)e^{jt} \right\} = \left( e^{-2t} - \cos t + 2\sin t \right), \quad t \geq 0. \quad (2.43) \]

The plot of \( y(t) \) is depicted in Fig.2.2.
b). The Laplace transform of the response signal has the form

\[ Y(s) = H(s)E(s) = \frac{s+1}{s^2+36} \frac{1}{s+2} = \frac{k_1}{s+2} + \frac{k_2}{s-j6} + \frac{k_3}{s+j6} \]  

(2.44)

where the coefficients are computed as follows

\[ k_1 = \lim_{s \to -2} \frac{s+1}{s^2+36} = -\frac{1}{40}, \]  

(2.45)

\[ k_2 = \lim_{s \to j6} \frac{s+1}{(s+j6)(s+2)} = \frac{3-j19}{240}, \]  

(2.46)

\[ k_3 = k_2^* = \frac{3+j19}{240}. \]  

(2.47)

Hence, we obtain the response

\[ y(t) = \mathcal{L}^{-1}\{Y(s)\} = -\frac{1}{40} e^{-2t} + 2\Re\left\{ \frac{3-j19}{240} e^{-j6t} \right\} = \left(-\frac{1}{40} e^{-2t} + \frac{1}{40}\cos 6t + \frac{19}{120}\sin 6t \right), \quad t \geq 0. \]  

(2.48)

The plot of \( y(t) \) is shown in Fig.2.3.

[Graph of y(t) shown]

**Question 2.9**

The impulse response of a linear time invariant system is

\[ h(t) = (2e^{-5t} + 3e^{-2t})u(t). \]  

(2.49)

Calculate the transfer function of this system.
Solution

Using the definition of the transfer function we write

\[ H(s) = \frac{Y(s)}{X(s)}, \quad (2.50) \]

where \( Y(s) \) is the Laplace transform of the zero-state response \( y(t) \) and \( X(s) \) is the Laplace transform of the input function \( x(t) \).

Since \( y(t) = h(t) = (2e^{-5t} + 3e^{-2t})u(t) \) and \( x(t) = \delta(t) \), then

\[ H(s) = \frac{Y(s)}{X(s)} = \frac{\mathcal{L}\{h(t)\}}{\mathcal{L}\{\delta(t)\}}. \quad (2.51) \]

Taking into account the equation

\[ \mathcal{L}\{\delta(t)\} = 1 \quad (2.52) \]

we find

\[ H(s) = \mathcal{L}\{h(t)\} = \mathcal{L}\{(2e^{-5t} + 3e^{-2t})u(t)\} = \frac{2}{s+5} + \frac{3}{s+2} = \frac{5s + 19}{(s+5)(s+2)}. \quad (2.53) \]

Question 2.10

The step response of a linear time invariant system is \( r(t) = (e^{-2t} + 3e^{-5t})u(t) \). Find the transfer function of this system.

Solution

The transfer function is

\[ H(s) = \frac{Y(s)}{X(s)}, \quad (2.54) \]

where

\[ Y(s) = R(s) = \mathcal{L}\{r(t)\} = \frac{1}{(s+2)^2} + \frac{3}{s+5} \quad (2.55) \]

and

\[ X(s) = \mathcal{L}\{x(t)\} = \mathcal{L}\{u(t)\} = \frac{1}{s}. \quad (2.56) \]

Setting (2.55) and (2.56) into (2.54) we obtain

\[ H(s) = R(s)s = \frac{s(3s^2 + 13s + 17)}{(s+2)^2(s+5)}. \quad (2.57) \]
**Question 2.11**

In a linear–time invariant system being in the zero–state the input $x(t)$ and the response $y(t)$ are as follows:

\[
x(t) = 3e^{-5t}u(t) \quad , \quad (2.58)
\]

\[
y(t) = (e^{-5t}\sin2t)u(t) \quad . \quad (2.59)
\]

Find the transfer function.

**Solution**

Using the definition of transfer function we find

\[
H(s) = \frac{Y(s)}{X(s)} = \mathcal{L}\{y(t)\} = \mathcal{L}\{e^{-5t}\sin2t\} = \mathcal{L}\{3e^{-5t}\} = \frac{2}{(s+5)^2 + 4} = \frac{2}{3} \frac{s + 5}{3s^2 + 10s + 29} \quad . \quad (2.60)
\]

**Question 2.12**

Using the convolution theorem find the inverse Laplace transform of the rational functions:

a) $F(s)=\frac{s^2}{(s^2+1)^3}$

b) $G(s)=\frac{1}{(s+1)(s^2+1)}$

**Solution**

a) Let

\[
F_1(s) = \frac{1}{s^2 + 1}, \quad F_2(s) = \frac{s}{s^2 + 1} \quad . \quad (2.61)
\]

We determine the inverse Laplace transforms of these functions:

\[
f_1(t) = \mathcal{L}^{-1}\{F_1(s)\} = (\sin t)u(t) \quad (2.62)
\]

\[
f_2(t) = \mathcal{L}^{-1}\{F_2(s)\} = (\cos t)u(t) \quad (2.63)
\]

and calculate the convolution of $f_1(t)$ and $f_2(t)$
\[ f(t) = \int_{-\infty}^{\infty} f_1(\tau)f_2(t-\tau)d\tau = \int_{-\infty}^{\infty} (\sin \tau)u(\tau)(\cos(t-\tau))u(t-\tau)d\tau = \int_{0}^{\infty} \sin \tau (\cos t \cos \tau + \sin t \sin \tau)d\tau = \cos \int_{0}^{\infty} \sin \tau \cos \tau d\tau + \sin \int_{0}^{\infty} \sin^2 \tau d\tau. \]  

We use the following formulas for undefined integrals:

\[ \int \sin x \cos x \, dx = \frac{1}{2} \sin^2 x, \quad (2.65) \]

\[ \int \sin^2 x \, dx = \frac{1}{2} x - \frac{1}{4} \sin 2x. \quad (2.66) \]

Hence,

\[ f(t) = \cos t \cdot \frac{1}{2} \sin^2 t + \sin \left( \frac{1}{2} t - \frac{1}{4} \sin 2t \right). \quad (2.67) \]

Applying the trigonometric identity

\[ \sin 2t = 2 \sin t \cos t, \quad (2.68) \]

we finally obtain

\[ f(t) = \frac{1}{2} \sin t. \quad (2.69) \]

b) Let us choose

\[ G_1(s) = \frac{1}{s+1}, \quad G_2(s) = \frac{1}{s^2 + 1} \quad (2.70) \]

and calculate the inverse Laplace transforms:

\[ g_1(t) = \mathcal{L}^{-1}\left( \frac{1}{s+1} \right) = e^{-t}u(t) \quad (2.71) \]

\[ g_2(t) = \mathcal{L}^{-1}\left( \frac{1}{s^2 + 1} \right) = (\sin t)u(t). \quad (2.72) \]

Using the convolution theorem we find:

\[ g(t) = \mathcal{L}^{-1}(G(s)) = \int_{-\infty}^{\infty} g_1(\tau)g_2(t-\tau)d\tau = \int_{-\infty}^{\infty} g_1(t-\tau)g_2(\tau)d\tau = \int_{-\infty}^{\infty} e^{-t}u(t-\tau)\sin \tau \, d\tau = e^{-t} \int_{0}^{t} e^{\tau} \sin \tau \, d\tau. \quad (2.73) \]

Now we apply the formula

\[ \int e^{\tau} \sin x \, d\tau = \frac{e^{\tau}}{2} (\sin x - \cos x) \quad (2.74) \]

and obtain

\[ g(t) = \frac{1}{2} e^{-t} \left[ e^{\tau} (\sin \tau - \cos \tau) \right]_0^{t} = \frac{1}{2} (\sin t - \cos t) + \frac{1}{2} e^{-t}, \quad t \geq 0. \quad (2.75) \]
3. Fourier Series

Question 3.1

Find a common period of the following signals:

a) \( f_1(t) = \sin 100t, \quad f_2(t) = \cos 500t, \)

b) \( f_1(t) = \sin \left( t + \frac{\pi}{6} \right), \quad f_2(t) = \cos 3t, \quad f_3(t) = \sin \left( \frac{t}{2} + \frac{\pi}{3} \right). \)

Solution

a) The periods of the functions \( f_1(t) \) and \( f_2(t) \) are

\[
T_1 = \frac{2\pi}{\omega_1} = \frac{2\pi}{100}, \quad T_2 = \frac{2\pi}{\omega_2} = \frac{2\pi}{500}.
\]

Since \( \frac{T_1}{T_2} = 5 \), \( T = T_1 = 5T_2 \) is the common period.

b) The periods of the functions \( f_1(t), f_2(t) \) and \( f_3(t) \) are as follows

\[
T_1 = \frac{2\pi}{1} = 2\pi, \quad T_2 = \frac{2\pi}{3}, \quad T_3 = \frac{2\pi}{1} = 4\pi.
\]

Since \( \frac{T_1}{T_2} = 3 \) the common period of \( f_1(t) \) and \( f_2(t) \) is \( T_{12} = T_1 = 3T_2 \).

Since \( \frac{T_{12}}{T_3} = \frac{1}{2} \) the common period is \( T_{123} = T_3 = 2T_{12} = 6T_2 = 4\pi \).

Question 3.2

Consider the periodic signal \( f(t) = \cos t \). Given \( h(x) = 2x^2 + 1 \), find the period of the function \( g(t) = h(f(t)) \).

Solution

Analytic description of the function \( g(t) \) is

\[
g(t) = 2\cos^2 t + 1.
\]

We use the trigonometric identity

\[
\cos^2 t = \frac{1}{2}(1 + \cos 2t)
\]
and rewrite function $g(t)$ as follows

$$g(t) = \cos 2t + 2.$$ 

The period of this function is the same as the period of $\cos 2t$. Hence, we have

$$T = \frac{2\pi}{2} = \pi.$$

**Question 3.3**

Calculate the $a_n$ and $b_n$ coefficients for the periodic functions shown in Fig.3.1. Find the Fourier series and plot the amplitude and phase spectra of the functions.

a) 

![Graph of function $f_1(t)$]

b) 

![Graph of function $f_2(t)$]
Solution

a) Since the function $f_1(t)$ is even its Fourier series does not contain the terms $b_n \sin n\omega_0 t$
and the coefficients $a_n$ ($n = 0, 1, 2, \ldots$) can be found using the following formulas

$$a_n = \frac{1}{T} \int_{-T/2}^{T/2} f_1(t) dt = \frac{2}{T} \int_0^{T/2} f_1(t) dt , \quad (3.1)$$

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} f_1(t) \cos n\omega_0 t dt = \frac{4}{T} \int_0^{T/2} f_1(t) \cos n\omega_0 t dt , \quad n = 1, 2, 3, \ldots \quad (3.2)$$

The period of $f_1(t)$ is $T = \frac{\pi}{50}$ and the angular frequency $\omega_0 = \frac{2\pi}{T} = 100 \text{ rad/s}$. Hence, we have

$$a_0 = \frac{100}{\pi} \int_0^{\pi/50} f_1(t) dt = \frac{100}{\pi} \int_{\pi/2}^{\pi/50} \cos 100t dt = \frac{1}{\pi} \sin \frac{\pi}{2} = \frac{1}{\pi} .$$

To calculate the other coefficients we apply formula (3.2)

$$a_n = \frac{200}{\pi} \int_0^{\pi/50} \cos 100t \cos n100t dt \quad n = 1, 2, 3, \ldots \quad (3.3)$$

Next we use the formula
\[ \int \cos ax \cos bx \, dx = \frac{\sin(a - b)x}{2(a - b)} + \frac{\sin(a + b)x}{2(a + b)}, \quad (3.4) \]

which is valid for \( a^2 \neq b^2 \) and the formula

\[ \int \cos^2 ax \, dx = \frac{x}{2} + \frac{\sin 2ax}{4a} \quad (3.5) \]

in order to calculate \( a_1 \). Hence, we have

\[ a_1 = \frac{200}{\pi} \left[ \cos^2 100t \right]_0^\pi = \frac{1}{2} \]

and on the basis of (3.3) and (3.4) we obtain \( a_n \) for \( n = 2,3,4,\ldots \)

\[ a_n = \frac{200}{\pi} \left[ \frac{\sin 100(1-n)t}{200(1-n)} + \frac{\sin 100(1+n)t}{200(1+n)} \right]_0^{\pi/2} = \frac{1}{\pi} \left[ \frac{\sin \left( \frac{1-n}{2} \pi \right)}{1-n} + \frac{\sin \left( \frac{1+n}{2} \pi \right)}{1+n} \right]. \]

The above formula leads to the following results

\[ a_2 = \frac{1}{\pi} \left[ \frac{\sin \left( -\frac{\pi}{2} \right)}{-1} + \frac{\sin \left( \frac{3\pi}{2} \right)}{3} \right] = \frac{2}{3\pi} \]

\[ a_3 = \frac{1}{\pi} \left[ \frac{\sin(-\pi)}{-2} + \frac{\sin(2\pi)}{4} \right] = 0 \]

\[ a_4 = \frac{1}{\pi} \left[ \frac{\sin \left( -\frac{3\pi}{2} \right)}{-3} + \frac{\sin \left( \frac{5\pi}{2} \right)}{5} \right] = \frac{-2}{15\pi}. \]

Hence, the Fourier series is

\[ f_1(t) = \frac{1}{\pi} + \frac{1}{2} \cos 100t + \frac{2}{3\pi} \cos 200t - \frac{2}{15\pi} \cos 400t + \ldots \]
Figure 3.2 shows the amplitude and phase spectra of \( f_1(t) \).

![Figure 3.2](image)

b) It is clear that \( f_2(t) = f_1(t - \frac{\pi}{200}) \). Thus, it holds

\[
f_2(t) = \frac{1}{\pi} + \frac{1}{2} \cos 100(t - \frac{\pi}{200}) + \frac{2}{3\pi} \cos 200(t - \frac{\pi}{200}) - \frac{2}{15\pi} \cos 400(t - \frac{\pi}{200}) + \ldots = \\
= \frac{1}{\pi} + \frac{1}{2} \cos \left(100t - \frac{\pi}{2}\right) - \frac{2}{3\pi} \cos 200t - \frac{2}{15\pi} \cos 400t + \ldots = \\
= \frac{1}{\pi} + \frac{1}{2} \sin 100t - \frac{2}{3\pi} \cos 200t - \frac{2}{15\pi} \cos 400t + \ldots
\]

Unlike \( f_1(t) \), the Fourier series expansion of function \( f_2(t) \) contains both the terms of the form \( a_n \cos n\omega_0 t \) and \( b_n \sin n\omega_0 t \). The amplitude spectrum is the same as in Fig.4.2a whereas the phases are \( \Theta_k = \Theta_k - k \frac{\pi}{2} \).

c) The period of \( f_3(t) \) is \( T = 2 \) whereas the angular frequency \( \omega_0 = \frac{2\pi}{T} = \pi \). Hence, we have

\[
a_0 = \frac{1}{T} \int_0^T f_3(t) dt = \frac{1}{2} \left[ \int_0^1 3 dt - \int_1^2 dt \right] = 1
\]

\[
a_n = \frac{2}{T} \int_0^T f_3(t) \cos n\omega_0 t dt = \frac{1}{3} \int_0^1 3 \cos n\pi t dt - \frac{2}{1} \int_1^2 \cos n\pi t dt = 3 \frac{1}{n\pi} \sin(n\pi) \bigg|_0^1 - \frac{1}{n\pi} \sin(n\pi) \bigg|_1^2 = 0
\]

\[
b_n = \frac{2}{T} \int_0^T f_3(t) \sin n\omega_0 t dt = \frac{1}{3} \int_0^1 3 \sin n\pi t dt - \frac{2}{1} \int_1^2 \sin n\pi t dt = -3 \frac{1}{n\pi} (\cos n\pi - 1) + \frac{1}{n\pi} (1 - \cos n\pi) = \\
= -\frac{4}{n\pi} \cos n\pi + \frac{4}{n\pi}.
\]
On the basis of these formulas we find

\[ b_1 = \frac{8}{\pi}, \quad b_2 = 0, \quad b_3 = \frac{8}{3\pi}, \quad b_4 = 0, \quad b_5 = \frac{8}{5\pi}, \quad \ldots \]

To plot the amplitude and phase spectra we find \( c_n = \sqrt{a_n^2 + b_n^2} = |b_n| \), \( \cos \Theta_n = 0 \), \( \sin \Theta_n = -1 \), hence, \( \Theta_n = -\frac{\pi}{2} \).

The spectra are shown in Fig.3.3.

**Question 3.4**

Calculate the \( a_n \) and \( b_n \) coefficients for the periodic functions shown in Fig.3.4. Find the Fourier series and plot the amplitude and phase spectra of these signals.

**a)**

**b)**
Solution

a) To represent the function of Fig.3.4a by a Fourier series we express the waveform as follows

\[
f_1(t) = \begin{cases} 
10 & \text{for } -0.5 < t < 0.5 \\
0 & \text{for } 0.5 < t \leq 1 \text{ and } -1 \leq t < -0.5. 
\end{cases} \quad (3.5)
\]

The period \( T = 2s \) and the angular frequency \( \omega_0 \) equals \( \frac{\pi}{s} \). The function satisfies the condition

\[
f_1(-t) = f_1(t), \quad (3.6)
\]

which means that it is an even function. The coefficients \( c_0 = a_0, a_n (b_n = 0, n = 1,2,3,..) \) are as follows:

\[
c_0 = \frac{1}{2} \int_{-0.5}^{0.5} 10t \, dt = 5t \bigg|_{-0.5}^{0.5} = 5,
\]

\[
a_n = \frac{2}{T} \int_{-0.5}^{0.5} 10\cos(n\pi t) \, dt = \frac{4}{T} \int_{0}^{0.5} 10\cos(n\pi t) \, dt = \frac{20}{n\pi} \sin(n\pi) \bigg|_{0}^{0.5} = \frac{20}{n\pi} \sin\left(\frac{n\pi}{2}\right). \quad (3.7)
\]

Hence,

\[
f_1(t) = 5 + \frac{20}{\pi} \cos\omega_0 t - \frac{20}{3\pi} \cos3\omega_0 t + \frac{20}{5\pi} \cos5\omega_0 t - \frac{20}{7\pi} \cos7\omega_0 t + \frac{20}{9\pi} \cos9\omega_0 t + \frac{20}{11\pi} \cos11\omega_0 t + ... \quad (3.8)
\]

The amplitude spectrum is presented in Fig.3.5 whereas, the phase spectrum is shown in Fig.3.6.
b) The function shown in Fig.3.4b is odd and has odd half wave symmetry property. We can express the waveform as follows \( T = 2s, \ \omega_0 = \frac{\pi \text{ rad}}{s} \):

\[
f_2(t) = \begin{cases} 
4t & \text{for } 0 \leq t \leq 0.5, \\
-4(t-1) & \text{for } 0.5 < t < 1.5, \\
4(t-2) & \text{for } 1.5 \leq t \leq 2.
\end{cases}
\]  

(3.9)

Taking into account the symmetry properties of the function we have

\[
a_n = 0, \quad n = 0, 1, 2, 3, \ldots \\
b_n = 0, \quad n = 2k, \ k = 1, 2, 3, \ldots
\]

(3.10)

We calculate the non-zero coefficients applying the reduced formulas, valid for functions having odd half wave symmetry property

\[
b_n = \frac{8}{T} \int_0^{0.5} 4t \sin(n \omega_0 t) \, dt = 16 \int_0^{0.5} t \sin(n \omega_0 t) \, dt = \\
= 16 \left[ -\frac{t}{n\pi} \cos(n\pi) + \frac{1}{n^2\pi^2} \sin(n\pi) \right]_0^{0.5} = 16 - \frac{1}{n^2\pi^2} \sin(n\frac{\pi}{2}), \quad n = 1, 3, 5, \ldots
\]

(3.11)

where we used the formula

\[
\int t \sin \alpha t \, dt = -\frac{t}{\alpha} \cos \alpha t + \frac{\sin \alpha t}{\alpha^2}.
\]

Hence, we obtain

\[
f_2(t) = 1.621 \sin \pi t - 0.18 \sin 3\pi t + 0.065 \sin 5\pi t - 0.033 \sin 7\pi t + \\
+ 0.02 \sin 9\pi t - 0.01 \sin 11\pi t + \ldots
\]

(3.12)

The amplitude and phase spectra are shown in Fig.3.7.
Question 3.5

Determine the exponential Fourier series of the signal shown in Fig.3.8, plot the amplitude and phase spectra.

Solution

The exponential Fourier series is
\[
f(t) = \sum_{n=-\infty}^{\infty} \tilde{c}_n e^{jn\omega_0 t},
\]
where
\[
\tilde{c}_n = \frac{1}{T} \int_{t_0}^{t_0+T} f(t) e^{-jn\omega_0 t} dt.
\]

Setting into (3.14) \(t_0=0\), taking into account the relationship \(f(t)=\frac{A}{T} t\) for \(0 < t < T\), and integrating by parts
\[
\int te^{-jn\omega_0 t} dt = \frac{te^{-jn\omega_0 t}}{-jn\omega_0} - \int \frac{e^{-jn\omega_0 t}}{-jn\omega_0} dt = \frac{te^{-jn\omega_0 t}}{-jn\omega_0} + \frac{e^{-jn\omega_0 t}}{n^2 \omega_0^2} = \frac{e^{-jn\omega_0 t}}{n^2 \omega_0^2} (jn\omega_0 t + 1),
\]
we have for \(n \neq 0\)
\[
\tilde{c}_n = \frac{1}{T} \left[ \int_{t_0}^{T} te^{-jn\omega_0 t} dt \right] = \frac{A}{T^2} \int_{t_0}^{T} e^{-jn\omega_0 t} dt = \frac{A}{T^2} \frac{e^{-jn\omega_0 t}}{n^2 \omega_0^2} (jn\omega_0 t + 1) \bigg|_{0}^{T} = \frac{\pi n A}{2}. \tag{3.15}
\]

For \(n = 0\) it holds
\[
\tilde{c}_0 = \frac{A}{T^2} \int_{0}^{T} t dt = A \frac{T^2}{2} \bigg|_{0}^{T} = \frac{A}{2}. \tag{3.16}
\]

On the basis of (3.15) we plot the amplitude and the phase spectra (Figs. 3.9 and 3.10).
Question 3.6

Repeat Question 3.5 for the signal depicted in Fig.3.11.

Solution

Function \( g(t) \) is obtained from \( f(t) \) shown in Fig.3.8 by a time displacement by \( \frac{T}{2} \). Hence, having the exponential Fourier series expansion of \( f(t) \)

\[
 f(t) = \sum_{n=-\infty}^{\infty} \tilde{c}_{n} e^{j n \omega_{0} t}
\]

we obtain

\[
 g(t) = f\left(t - \frac{T}{2}\right) = \sum_{n=-\infty}^{\infty} \tilde{c}_{n} e^{-j (n - \frac{1}{2}) \omega_{0} t} = \sum_{n=-\infty}^{\infty} \tilde{c}_{n} e^{j n \omega_{0} t},
\]

where
\[ \tilde{c}_n = \tilde{c}_n e^{-j\omega_0 \tau} = \tilde{c}_n e^{-jr \frac{2\pi}{T} \tau} = \tilde{c}_n e^{-jn\pi}. \]

Using the coefficients \( \tilde{c}_n \) specified in Question 3.5 we find:

\[ \tilde{c}_0 = \tilde{c}_0 = \frac{A}{2}, \quad |\tilde{c}_0| = \frac{A}{2}, \]

\[ \tilde{c}_n = \frac{fA}{2\pi n} e^{-jn\pi} = \frac{A}{2\pi n} e^{j\left( \frac{\pi}{2} - n\pi \right)}, \quad |\tilde{c}_n| = \frac{A}{2\pi |n|}, \quad \varphi_n = \begin{cases} \frac{\pi}{2} - n\pi & \text{for } n > 0 \\ \frac{\pi}{2} - n\pi & \text{for } n < 0. \end{cases} \]

The obtained results implicate that the amplitude spectrum of \( g(t) \) is the same as \( f(t) \), whereas the phases are different as it is shown in Fig.3.12 where the phases are reduced to the interval \([-\pi/2, \pi/2]\).

![Fig.3.12](image)

**Question 3.7**

Determine the exponential Fourier series of the periodic signal described by the formula

\[ f(t) = \begin{cases} \sin(100t) & \text{for } 0 \leq t \leq \frac{\pi}{50} \\ 0 & \text{for } \frac{\pi}{50} \leq t \leq \frac{\pi}{25}. \end{cases} \tag{3.17} \]

where \( T = \frac{\pi}{25} \).
Solution

Applying integration by parts:
\[
\int \sin 100t e^{-j\omega_0 t} dt = -\frac{e^{-j\omega_0 t}}{100} \cos 100t - \frac{jn\omega_0}{100} \int \cos 100t e^{-j\omega_0 t} dt =
\]
\[
-\frac{e^{-j\omega_0 t}}{100} \cos 100t - \frac{jn\omega_0}{100^2} e^{-j\omega_0 t} \sin 100t + \frac{n^2 \omega_0^2}{100^2} \int \sin 100t e^{-j\omega_0 t} dt ,
\]

\[
\left(1 - \frac{n^2 \omega_0^2}{100^2}\right) \int \sin 100t e^{-j\omega_0 t} dt = -\frac{e^{-j\omega_0 t}}{100} \cos 100t - \frac{jn\omega_0}{100^2} e^{-j\omega_0 t} \sin 100t ,
\]

\[
\int \sin 100t e^{-j\omega_0 t} dt = \frac{e^{-j\omega_0 t}}{100^2 - n^2 \omega_0^2} (-jn\omega_0 \sin 100t - 100 \cos 100t) ,
\]

we find

\[
\tilde{c}_n = \frac{1}{T} \int_0^T \sin 100t e^{-j\omega_0 t} dt = \frac{1}{T} \frac{e^{-j\omega_0 t}}{100^2 - n^2 \omega_0^2} (-jn\omega_0 \sin 100t - 100 \cos 100t) \bigg|_0^{T/2} =
\]

\[
= \frac{25}{\pi 50^2 (4 - n^2)} 100 (1 - \cos \pi) ,
\]

Hence, we have

\[
\tilde{c}_0 = 0, \quad \tilde{c}_n = \frac{2}{\pi (4 - n^2)} \quad n = 1, 3, 5, \ldots.
\]

The amplitude spectrum is plotted in Fig.3.13 whereas the phase spectrum is presented in Fig.3.14.

![Fig.3.13](image-url)
Question 3.8

Determine the effective values of the signals shown in Fig.3.15.
Solution

a) According to the definition

\[ F_{\text{eff}} = \sqrt{\frac{1}{T} \int_0^T f_1^2(t) \, dt} = 10. \]

b) Similarly we obtain
\[
F_{2\text{eff}} = \sqrt{\frac{1}{T} \int_0^T f_2^2(t) \, dt} = \sqrt{\frac{1}{0.8} \int_0^{0.8} 25 \, dt} = \sqrt{0.8 \cdot 25} = 5\sqrt{0.8} = 4.47.
\]

c)
\[
F_{3\text{eff}} = \sqrt{\frac{1}{T} \int_0^T f_3^2(t) \, dt} = \sqrt{\frac{1}{0.5} \int_0^{0.5} (6t)^2 \, dt + \int_0^{0.5} 9 \, dt} = \sqrt{\frac{1}{3} \cdot 36 \cdot (0.5)^3 + 4.5} = \sqrt{6} = 2.45.
\]

d)
\[
f_{4\text{eff}} = \sqrt{\frac{1}{T} \int_0^T f_4^2(t) \, dt} = \sqrt{\int_0^{0.5} \sin^2(2\pi t) \, dt} = \sqrt{\int_0^{0.5} \left(\frac{t}{2} - \frac{\sin(8\pi t)}{8\pi}\right)^2 \, dt} = \sqrt{\frac{1}{4} - \frac{1}{2}} = \sqrt{\frac{1}{4}} = \frac{1}{2}.
\]

e) The effective value of the function \(f_5(t)\) is the same as the effective value of sine wave, hence, it holds
\[
F_{5\text{eff}} = \frac{1}{\sqrt{2}}.
\]
4. The Fourier transform

Question 4.1

The Fourier transform $F(j\omega)$ of a time function is shown in Fig.4.1. Find the time function $f(t)$.

![Fig.4.1](image)

Solution

We compute the inverse Fourier transform of $F(j\omega)$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\epsilon} A e^{j\omega t} d\omega = A \frac{e^{j\omega t}}{\pi t} \left|_{\epsilon}^{\infty} \right. = \frac{A}{\pi t} \sin (\epsilon t) = \frac{2\epsilon A}{\pi} \sin (\epsilon t)$$

Question 4.2

Given signal $f(t)$

$$f(t) = \begin{cases} 2e^{-2t}, & 0 < t < 1, \\ 0, & \text{otherwise}. \end{cases} \quad (4.1)$$

Introduce a periodic signal $g(t)$ that coincides with $f(t)$ for $0 < t < 1$ and express it by the exponential Fourier series. Compare with the Fourier transform of $f(t)$.

Solution

We can choose the function $g(t)$ as shown in the Fig.4.2, where the periodic function is described by the equation:

$$g(t) = 2e^{-2t}, \quad 0 < t < T, \quad T = 1s. \quad (4.2)$$

The exponential Fourier series has the form
\[
g(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn \omega_0 t}, \quad (4.3)
\]

where
\[
c_n = \frac{1}{T} \int_{t_0}^{t_0+T} g(t) e^{-jn \omega_0 t} dt \quad (4.4)
\]

and \( T = 1, \omega_0 = 2\pi \).
Assuming \( t_0 = 0 \) we obtain
\[
\tilde{c}_n = \int_{0}^{1} 2e^{-2t} e^{-jn \omega_0 t} dt = -\left. \frac{2}{2 + jn \omega_0} e^{-t(2+jn \omega_0)} \right|_0^1 = \left. \frac{2}{2 + jn \omega_0} (1 - e^{-(2+jn \omega_0)}) \right. \quad (4.5)
\]

The Fourier transform of the function \( f(t) \) has the form
\[
F(j \omega) = \int_{-\infty}^{\infty} f(t) e^{-j \omega t} dt = \int_{0}^{1} 2e^{-t(2+j \omega)} dt = -\left. \frac{2}{2 + j \omega} e^{-t(2+j \omega)} \right|_0^1 = \frac{2}{2 + j \omega} (1 - e^{-(2+j \omega)}) \quad (4.6)
\]

Comparing (4.5) and (4.6) we find that
\[
T \tilde{c}_n = F(j \omega) \bigg|_{\omega = n \omega_0}.
\]

**Question 4.3**

Find the Fourier transform of the signal \( f(t) \) defined over an interval of length 2
\[
f(t) = \begin{cases} 2 & \text{for } 0 < t < 1 \\ 1 & \text{for } 1 < t < 2 \end{cases} \quad (4.7)
\]

and equal to zero outside this interval.
Form a periodic signal \( g(t) \) that coincides with \( f(t) \) for \( 0 < t < 2 \) and expand it into the exponential Fourier series. Compare with the Fourier transform of \( f(t) \).

**Solution**

Firstly, we find the Fourier transform of the signal \( f(t) \)

\[
F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = 2 \int_{0}^{1} e^{-j\omega t} dt + \int_{1}^{2} e^{-j\omega t} dt = \frac{2 j e^{-j\omega}}{\omega} \left|_0^1 + \frac{1}{\omega} e^{-j\omega} \right|^2 = \frac{j}{\omega} (e^{-j\omega} + e^{-2j\omega} - 2) = \frac{1}{\omega} e^{j\pi/2} [\cos \omega + \cos 2\omega - 2 - j(\sin \omega + \sin 2\omega)].
\]  

Hence, we obtain

\[
|F(j\omega)| = \frac{1}{|\omega|} \sqrt{(\cos \omega + \cos 2\omega - 2)^2 + (\sin \omega + \sin 2\omega)^2},
\]  

\[
\angle F(j\omega) = \frac{\pi}{2} + \tan^{-1} \left( \frac{-\sin \omega + \sin 2\omega}{\cos \omega + \cos 2\omega - 2} \right).
\]

We introduce the function \( g(t) \) that coincides with \( f(t) \) for \( 0 < t < 2 \) as shown in the Fig.4.3.

In this case

\[
g_2(t) = \sum_{n=-\infty}^{\infty} \tilde{c}_n e^{j n\omega_0 t}, \quad T = 4s, \quad \omega_0 = \frac{\pi}{2},
\]

where

\[
\tilde{c}_n = \frac{1}{T} \int_{0}^{T} g_2(t) e^{-j n\omega_0 t} dt = \frac{1}{4} \int_{0}^{1} 2 e^{-j n\omega_0 t} dt + \frac{1}{4} \int_{1}^{2} e^{-j n\omega_0 t} dt = \frac{j}{4n\omega_0} (e^{-j n\omega_0} + e^{-j 2n\omega_0} - 2) = \frac{e^{j\pi/2}}{4n\omega_0} [\cos n\omega_0 + \cos 2n\omega_0 - 2 - j(\sin n\omega_0 + \sin 2n\omega_0)].
\]

Comparing (4.12) with the Fourier transform of \( f(t) \) (4.8) we observe, that

\[
T\tilde{c}_n = F(j\omega) \bigg|_{\omega = \pi / \omega_0}.
\]
Question 4.4

Find the Fourier transforms of the following functions:

a) \( f_1(t) = 2e^{-|t|} \quad -\infty < t < \infty \), \hspace{1cm} (4.13)

\[
\begin{cases}
0 & \text{for } t < 0 \\
1 & \text{for } 0 < t < 1
\end{cases}
\]

b) \( f_2(t) = \begin{cases}
2 & \text{for } 1 < t < 2 \\
3 & \text{for } 2 < t < 3 \\
0 & \text{for } t > 3
\end{cases} \), \hspace{1cm} (4.14)

\[
\begin{cases}
0 & \text{for } t < 0 \\
2t & \text{for } 0 < t < 1 \\
0 & \text{for } t > 1
\end{cases}
\]

c) \( f_3(t) = \begin{cases}
0 & \text{for } t < 0 \\
2t & \text{for } 0 < t < 1 \\
0 & \text{for } t > 1
\end{cases} \), \hspace{1cm} (4.15)

Solution

a) \( F(j\omega) = \int_{-\infty}^{\infty} 2e^{-|t|}e^{-j\omega t} \, dt = \int_{0}^{\infty} 2e^{-t}e^{-j\omega t} \, dt + \int_{0}^{\infty} 2e^{-t}e^{-j\omega t} \, dt = \)

\[
= \frac{2}{1-j\omega} \left[ e^{-(1+j\omega)t} \right]_{0}^{\infty} - \frac{2}{1+j\omega} \left[ e^{-(1+j\omega)t} \right]_{0}^{\infty} = \frac{2}{1-j\omega} + \frac{2}{1+j\omega} = \frac{4}{1+\omega^2}. \hspace{1cm} (4.16)
\]

b) The function \( f_2(t) \) is shown in Fig.4.4.

![Fig.4.4](image-url)
Let us introduce an auxiliary function

\[
g(t) = \begin{cases} 
1 & \text{for } 0 < t < 1, \\
0 & \text{for } t > 1 \text{ or } t < 0, 
\end{cases}
\] (4.17)

shown in Fig.4.5. The Fourier transform of this function is

\[
G(j\omega) = \int_0^1 e^{-j\omega t} \, dt = \frac{j}{\omega} \left( e^{-j\omega} - 1 \right).
\] (4.18)

Function \(f_2(t)\) can be expressed in terms of \(g(t)\) as follows

\[
f_2(t) = g(t) + 2g(t - t_1) + 3g(t - t_2),
\] (4.19)

where \(t_1 = 1s, t_2 = 2s\).

Applying the linearity and shifting properties of the Fourier transform we obtain

\[
F_2(j\omega) = G(j\omega) + 2G(j\omega)e^{-j\omega} + 3G(j\omega)e^{-j2\omega} =
\]

\[
= \frac{j}{\omega} \left[ e^{-j\omega} - 1 + 2(e^{-j\omega} - 1)e^{-j\omega} + 3(e^{-j\omega} - 1)e^{-j2\omega} \right] =
\]

\[
= \frac{j}{\omega} \left[ 3e^{-j2\omega} - e^{-j2\omega} - e^{-j\omega} - 1 \right].
\] (4.20)

The Fourier transform \(F_2(j\omega)\) can be also found directly, using the definition formula

\[
F_2(j\omega) = \int_0^1 e^{-j\omega t} \, dt + \int_1^2 2e^{-j\omega t} \, dt + \int_2^3 3e^{-j\omega t} \, dt =
\]

\[
= -\frac{1}{\jmath \omega} e^{-j\omega t} \bigg|_0^1 - \frac{2}{\jmath \omega} e^{-j\omega t} \bigg|_1^2 - \frac{3}{\jmath \omega} e^{-j\omega t} \bigg|_2^3 =
\]

\[
= -\frac{1}{\jmath \omega} \left( e^{-j\omega} - 1 + 2e^{-j2\omega} - 2e^{-j\omega} + 3e^{-j3\omega} - 3e^{-j2\omega} \right) =
\]

\[
= \frac{j}{\omega} \left( 3e^{-j3\omega} - e^{-j2\omega} - e^{-j\omega} - 1 \right).
\]
c) The Fourier transform of function $f_3(t)$ described by the formula (4.15) and shown in Fig.4.6 is

$$F_3(j\omega) = \frac{1}{j\omega} \left[ 2te^{-j\omega t} dt \right].$$

(4.21)

Firstly, we find $\int te^{-j\omega t} dt$ integrating by parts:

$$u = t, \quad dv = e^{-j\omega t} dt, \quad du = dt, \quad v = -\frac{1}{j\omega} e^{-j\omega t},$$

$$\int te^{-j\omega t} dt = uv - \int vdu = -\frac{t}{j\omega} e^{-j\omega t} + \int \frac{1}{j\omega} e^{-j\omega t} dt = -\frac{t}{j\omega} e^{-j\omega t} + \frac{1}{\omega} e^{-j\omega t}. \quad (4.22)$$

Using (4.21) we obtain

$$F_3(j\omega) = \left[ 2te^{-j\omega t} dt \right] = 2 \left[ -\frac{t}{j\omega} e^{-j\omega t} + \frac{1}{\omega} e^{-j\omega t} \right] _0^1 = 2 \left[ -\frac{e^{-j\omega}}{j\omega} + \frac{e^{-j\omega}}{\omega^2} - \frac{1}{\omega^2} \right] =$$

$$= 2e^{-j\omega} \left( -\frac{1}{j\omega} + \frac{1}{\omega^2} \right) - \frac{2}{\omega^2}. \quad (4.23)$$

Using the result obtained for the function $f_3(t)$ in Question 4.4c find the Fourier transform of the signal depicted in Fig.4.7.

**Question 4.5**

This signal can be expressed in the form

$$f(t)$$

Fig.4.7

**Solution**
\[ f(t) = f_3(t + 1) + f_3(-t + 1). \]

To find \( \mathcal{F}(f_3(t + 1)) \) we apply the shifting property

\[ \mathcal{F}(f_3(t + 1)) = F_3(j\omega)e^{j\omega}. \quad (4.24) \]

Using this result and the scaling property we obtain

\[ \mathcal{F}(f_3(-t + 1)) = F_3(-j\omega)e^{-j\omega}. \quad (4.25) \]

We combine the results (4.24) and (4.25) and substitute \( F_3(j\omega) \) given by (4.23)

\[
F(j\omega) = \left[ 2e^{-j\omega}\left(-\frac{1}{j\omega} + \frac{1}{\omega^2}\right) - \frac{2}{\omega^2}\right]e^{j\omega} + \left[ 2e^{j\omega}\left(\frac{1}{j\omega} + \frac{1}{\omega^2}\right) - \frac{2}{\omega^2}\right]e^{-j\omega} = \\
\frac{4}{\omega^2} - \frac{2}{\omega^2}(e^{j\omega} + e^{-j\omega}) = \frac{4}{\omega^2}(1 - \cos\omega). \quad (4.26)
\]

**Question 4.6**

The Fourier transform of a signal \( x(t) \) is given

\[ X(j\omega) = \frac{1}{1 + j\omega}. \quad (4.27) \]

Find the Fourier transforms of the following signals:

a) \( x_1(t) = 2x(t) + 3x(t - 1). \)

b) \( x_2(t) = x\left(-\frac{t}{2}\right) + \frac{dx}{dt}. \)

c) \( x_3(t) = e^{-3t}x(-1-t). \)

d) \( x_4(t) = 5x(t)\cos 3t. \)

**Solution**

a) Using linearity and shifting properties we obtain

\[
X_1(j\omega) = 2X(j\omega) + 3X(j\omega)e^{-j\omega} = \frac{2}{1 + j\omega} + 3\frac{1}{1 + j\omega}e^{-j\omega} = \frac{2 + 3e^{-j\omega}}{1 + j\omega}. \quad (4.28)
\]

b) Applying scaling and differentiation rules we have

\[
X_2(j\omega) = -2X(-2j\omega) + j\omega X(j\omega) = \frac{2}{1 - 2j\omega} + \frac{j\omega}{1 + j\omega}. \quad (4.29)
\]
c) First, we apply time shifting property to function \( x(t) \)

\[ x(t - 1) \rightarrow X(j \omega)e^{-j\omega}. \]

From the scaling property we obtain

\[ x(-t - 1) \rightarrow X(-j \omega)e^{j\omega}. \]

Finally, applying frequency shifting property we find

\[ X_3(j \omega) = X(-j(\omega + 3))e^{j(\omega + 3)} = \frac{1}{1 - j(\omega + 3)}e^{j(\omega + 3)}. \]

d) Function \( x_4(t) \) can be expressed in terms of exponential functions as follows

\[ x_4(t) = 5x(t)\frac{e^{j3t} + e^{-j3t}}{2}. \]

Thus, applying frequency shifting property we obtain:

\[ X_4(j \omega) = \frac{5}{2}\left[ \frac{1}{1 + j(\omega - 3)} + \frac{1}{1 + j(\omega + 3)} \right]. \quad (4.30) \]

**Question 4.7**

Determine directly the Fourier transform of the half cosine pulse shown in Fig.4.8.

![Fig.4.8](image_url)

Using this result determine the Fourier transforms of the signals depicted in Figs.4.9, 4.10.

![Fig.4.9](image_url)
Solution

The Fourier transform of \( f(t) \) is given by

\[
F(j\omega) = \int_{-T/4}^{T/4} \cos(\omega_0 t) \cdot e^{-j\omega t} \, dt ,
\]  

(4.31)

where

\[
\omega_0 = \frac{2\pi}{T}.
\]

Setting

\[
\cos(\omega_0 t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}
\]

we find

\[
F(j\omega) = \frac{1}{2} \left( \int_{-T/4}^{T/4} e^{-j(\omega_0 t)} + e^{-j(\omega + \omega_0) t} \right) dt = \]

\[
= \frac{1}{2} \left( \left. e^{-j(\omega_0 t)} \right|_{-T/4}^{T/4} + \left. e^{-j(\omega + \omega_0) t} \right|_{-T/4}^{T/4} \right) = \]

\[
= \frac{j}{2(\omega - \omega_0)} \left( e^{-j(\omega_0 t)} \frac{T}{4} - e^{j(\omega_0) t} \frac{T}{4} \right) + \frac{j}{2(\omega + \omega_0)} \left( e^{-j(\omega + \omega_0) t} \frac{T}{4} - e^{j(\omega + \omega_0) t} \frac{T}{4} \right) = \]

\[
= \frac{1}{\omega - \omega_0} \sin(\omega - \omega_0) \frac{T}{4} + \frac{1}{\omega + \omega_0} \sin(\omega + \omega_0) \frac{T}{4} .
\]

Function \( x(t) \) can be specified as follows:

\[
x(t) = -f(t) - f\left( t + \frac{T}{2} \right) - f\left( t - \frac{T}{2} \right) .
\]  

(4.33)
Using the linearity property and the time shifting theorem we obtain

\[
X(j\omega) = -F(j\omega) - F(j\omega)e^{j\omega \frac{T}{2}} - F(j\omega)e^{-j\omega \frac{T}{2}} =
\]

\[
= -F(j\omega) \left(1 + e^{j\omega \frac{T}{2}} + e^{-j\omega \frac{T}{2}}\right) = -F(j\omega) \left(1 + 2\cos \omega \frac{T}{2}\right) =
\]

\[
= \left[-\frac{1}{\omega - \omega_0}\sin(\omega - \omega_0)\frac{T}{4} + \frac{1}{\omega + \omega_0}\sin(\omega + \omega_0)\frac{T}{4}\right] \left(1 + 2\cos \omega \frac{T}{2}\right).
\]

Signal \(y(t)\) can be expressed in terms of \(f(t)\) as follows

\[
y(t) = f\left(t - \frac{T}{4}\right) - f\left(t - \frac{3}{4}T\right).
\]

Hence, it holds

\[
Y(j\omega) = F(j\omega)e^{-j\omega \frac{T}{4}} - F(j\omega)e^{-j\omega \frac{3}{4}T} = F(j\omega)\left(e^{-j\omega \frac{T}{4}} - e^{-j\omega \frac{3}{4}T}\right) =
\]

\[
= \left[-\frac{1}{\omega - \omega_0}\sin(\omega - \omega_0)\frac{T}{4} + \frac{1}{\omega + \omega_0}\sin(\omega + \omega_0)\frac{T}{4}\right] \left(e^{-j\omega \frac{T}{4}} - e^{-j\omega \frac{3}{4}T}\right).
\]

**Question 4.8**

Determine directly the Fourier transform of the signal

\[
x(t) = \begin{cases} \cos \omega_0 t & \text{for } -\frac{3}{4}T < t < \frac{3}{4}T, \\ 0 & \text{otherwise}, \end{cases}
\]

where \(T = \frac{2\pi}{\omega_0}\).

\[
\text{Fig.4.11}
\]

**Solution**

To determine the Fourier transform of the signal \(x(t)\) we use the Fourier integral
\[
X(j \omega) = \frac{3}{4}e^{-j\omega t} \left[ \frac{1}{2} \left( e^{j\omega t} + e^{-j\omega t} \right) e^{-j\omega t} \right] dt = \frac{1}{2} \left[ \frac{3}{4}e^{-j(\omega+\omega_0)\frac{3T}{4}} - \frac{3}{4}e^{-j(\omega-\omega_0)\frac{3T}{4}} \right] \]

\[
\text{Question 4.9}
\]

The carrier signal is given by

\[
v_c(t) = 12 \cos \omega_0 t,
\]
where \( \omega_0 = 2\pi 10^4 \text{ rad/s} \). The message signal is given by

\[
v_m(t) = \cos \omega'_0 t + 2\cos 3\omega'_0 t,
\]
where \( \omega'_0 = 2\pi 10^2 \text{ rad/s} \).

Determine and sketch the spectrum of the signal

\[
g(t) = v_m(t)v_c(t).
\]

\section*{Solution}

The signal \( g(t) \) is as follows

\[
g(t) = (\cos \omega'_0 t + 2\cos 3\omega'_0 t)12 \cos \omega_0 t = g_1(t) + g_2(t),
\]
where

\[
g_1(t) = 12 \cos 2\pi 10^2 t \cdot \cos 2\pi 10^4 t,
\]

\[
g_2(t) = 24 \cos 6\pi 10^2 t \cdot \cos 2\pi 10^4 t.
\]

Using the modulation property we write

\[
\mathcal{F}(g_1(t)) = 6F_i(j(\omega - 2\pi 10^4)) + 6F_i(j(\omega + 2\pi 10^4)),
\]

\[131\]
where
\[ F_1(j \omega) = \mathcal{F}\left(\cos 2\pi t^2 t\right) = \pi \left(\delta(\omega - 2\pi 10^2) + \delta(\omega + 2\pi 10^2)\right). \] (4.46)

Thus,
\[ \mathcal{F}(g_1(t)) = 6\pi \left[\delta(\omega - 2\pi 10^2 - 2\pi 10^4) + \delta(\omega + 2\pi 10^2 - 2\pi 10^4) + \delta(\omega - 2\pi 10^2 + 2\pi 10^4) + \delta(\omega + 2\pi 10^2 + 2\pi 10^4)\right] \] (4.47)
holds.

Similarly we obtain
\[ \mathcal{F}(g_2(t)) = 12\pi \left[\delta(\omega - 6\pi 10^2 - 2\pi 10^4) + \delta(\omega + 6\pi 10^2 - 2\pi 10^4) + \delta(\omega - 6\pi 10^2 + 2\pi 10^4) + \delta(\omega + 6\pi 10^2 + 2\pi 10^4)\right] \] (4.48)
and the spectrum of \( g(t) \) is
\[ G(j \omega) = \mathcal{F}(g_1(t)) + \mathcal{F}(g_2(t)). \] (4.49)

It is depicted in Fig.4.12

**Question 4.11**

Verify Parseval’s theorem for the signal
\[ x(t) = \begin{cases} e^{at} & t \leq 0, \\ 0 & t > 0, \end{cases} \] (4.50)
where \( a \) is a real positive number.

**Solution**

Firstly, we determine the Fourier transform of \( x(t) \)
\[ X(j\omega) = \int_{-\infty}^{0} e^{at} e^{-j\omega t} dt = \frac{1}{a-j\omega} e^{(a-j\omega)t} \bigg|^{0}_{-\infty} = \frac{1}{a-j\omega} . \] (4.51)

The energy \( E \) of signal \( x(t) \) is given by the formula

\[ E = \int_{-\infty}^{\infty} x^2(t) dt = \int_{-\infty}^{0} e^{2at} dt = \frac{1}{2a} e^{2at} \bigg|^{0}_{-\infty} = \frac{1}{2a} . \] (4.52)

Now we calculate the energy expressed in the frequency domain by the formula

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{a^2 + \omega^2} d\omega . \] (4.53)

To compute the integral on the right hand side of (4.53) we write

\[ \int_{-\infty}^{\infty} \frac{1}{a^2 + \omega^2} d\omega = \frac{1}{a^2} \int_{-\infty}^{\infty} \frac{1}{a^2} \frac{1}{1+\left(\frac{\omega}{a}\right)^2} d\omega = \frac{1}{a} \int_{-\infty}^{\infty} \frac{1}{\frac{\omega}{a}} d\omega = \frac{1}{a} \tan^{-1}\left(\frac{\omega}{a}\right) \bigg|^{\infty}_{-\infty} = \frac{1}{a} \left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \frac{\pi}{a} . \] (4.54)

Substituting (4.54) into (4.53) we find

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = \frac{1}{2a} . \]

The obtained result confirms Parseval’s theorem.
5. Discrete Fourier Transform (DFT) and Fast Fourier Transform (FFT)

Question 5.1

Given the following sequences:

a) \{1,1,0,0\}

b) \{1,2,3,4\},

find the DFT of the sequences and verify the results using the inverse DFT.

Solution

a) The DFT is defined by the equation

\[ F_n = \sum_{m=0}^{N-1} f_m w^{-mn} \quad n = 0,1,2,\ldots,N-1, \quad (5.1) \]

where

\[ w = e^{-\frac{2\pi}{N}}. \quad (5.2) \]

In the considered case \( N = 4 \), hence, we have:

\[ w = e^{\frac{j\pi}{2}} = j, \]

\[ F_0 = \sum_{m=0}^{1} f_m = 2, \]

\[ F_1 = \sum_{m=0}^{1} f_m w^{-m} = 1 + e^{-\frac{j\pi}{2}} = 1 + j, \]

\[ F_2 = \sum_{m=0}^{1} f_m w^{-2m} = 1 + e^{-j\pi} = 0, \]

\[ F_3 = \sum_{m=0}^{1} f_m w^{-3m} = 1 + e^{-\frac{3j\pi}{2}} = 1 - j. \]

Thus, the DFT of the sequence a) is the sequence

\[ \{F_n\} = \{2, 1-j, 0, 1+j\}. \]

To verify the result we use the inverse DFT

\[ f_m = \frac{1}{N} \sum_{n=0}^{N-1} F_n w^{mn}, \quad m = 0,1,\ldots,N-1. \quad (5.3) \]

Substituting \( N = 4 \) and \( w \) given by (5.2) we obtain
$$f_m = \frac{1}{4} \sum_{n=0}^{3} F_n e^{\frac{j\pi mn}{4}} , \quad m = 0, 1, 2, 3 \ldots$$

Hence, it holds:

$$f_0 = \frac{1}{4} \sum_{n=0}^{3} F_n = 1.$$  

$$f_1 = \frac{1}{4} \sum_{n=0}^{3} F_n e^{j\frac{\pi}{2}} = \frac{1}{4} (2 + 1 - j + 1) = 1.$$  

$$f_2 = \frac{1}{4} \sum_{n=0}^{3} F_n e^{j\pi n} = \frac{1}{4} (2 - 1 - j - j) = 0.$$  

$$f_3 = \frac{1}{4} \sum_{n=0}^{3} F_n e^{j\frac{3\pi}{2}} = \frac{1}{4} (2 - j + 1 + 1) = 0.$$  

b) Setting $N = 4$ and $w = e^{j\frac{\pi}{2}}$ into (5.1) we have

$$F_n = \sum_{m=0}^{3} f_m e^{-j\frac{\pi mn}{2}} , \quad n = 0, 1, 2, 3 \ldots$$

Hence, the following equations hold. To verify these results we apply the inverse DFT formula (5.3)

$$f_m = \frac{1}{4} \sum_{n=0}^{3} F_n e^{\frac{j\pi mn}{4}} , \quad m = 0, 1, 2, 3.$$  

Setting $m = 0, 1, 2, 3$ we find:

$$f_0 = \frac{1}{4} \left(10 - 2 + j2 - 2 - j2\right) = 1.$$  

$$f_1 = \frac{1}{4} \left(10 + (-2 + j2)e^{j\frac{\pi}{2}} - 2e^{j\pi} + (-2 - j2)e^{j\frac{3\pi}{2}}\right) = 2.$$  

$$f_2 = \frac{1}{4} \left(10 + (-2 + j2)e^{j\pi} - 2e^{j2\pi} + (-2 - j2)e^{j3\pi}\right) = 3.$$  

$$f_3 = \frac{1}{4} \left(10 + (-2 + j2)e^{j\frac{3\pi}{2}} - 2e^{j3\pi} + (-2 - j2)e^{j\frac{9\pi}{2}}\right) = 4.$$  

**Question 5.2**
For the sequence \(\{1, 1, 1, 1, 1, 1, 0, 0, 0\}\) find the DFT and plot the magnitude and phase spectra.

**Solution**

For given sequence \(N = 9\) and

\[ w = e^{\frac{2\pi}{N}} = e^{\frac{2\pi}{9}}. \]

Hence, we have

\[
F_n = \sum_{m=0}^{N-1} f_m w^{-mn} = \sum_{m=0}^{5} e^{-j\frac{2\pi}{9} mn}, \quad n = 0, 1, \ldots, 8. 
\]

Setting \(n = 0, 1, \ldots, 8\) we find:

\[
F_0 = 6, \quad |F_0| = 6, \quad \angle F_0 = 0^\circ, \\
F_i = 1 + e^{-j\frac{2\pi}{9}} + e^{-j\frac{4\pi}{9}} + e^{-j\frac{6\pi}{9}} + e^{-j\frac{8\pi}{9}} + e^{-j\frac{10\pi}{9}}. 
\]

The expression on the right hand side is a geometric series of the form

\[
a_i + a_1q + a_1q^2 + \ldots + a_1q^{\nu-1},
\]

where \(\nu = 6, a_1 = 1, q = e^{-j\frac{2\pi}{9}}\).

The sum of this series is

\[
\sum_{l=0}^{\nu-1} a_1q^l = \begin{cases} 
   a_1 \dfrac{1-q^\nu}{1-q} & \text{for } q \neq 1, \\
   \nu a_1 & \text{for } q = 1.
\end{cases}
\]

Hence, we obtain

\[
F_i = 1 - e^{-j\frac{2\pi}{9}} = 1 - e^{-j\frac{4\pi}{9}} = 1 - \cos\frac{4\pi}{3} + j\sin\frac{4\pi}{3} = 1.5 - j0.866 = 2.53e^{-j0.51}\nu.
\]

Thus, we have

\[
|F_i| = 2.53, \quad \angle F_i = -100^\circ.
\]

Similarly we find the other numbers of the DFT, listed below:
\[ |F_2| = 1.35 \quad \angle F_2 = -20^\circ \]
\[ |F_3| = 0 \]
\[ |F_4| = 0.88 \quad \angle F_4 = -140^\circ \]
\[ |F_5| = 0.88 \quad \angle F_5 = 140^\circ \]
\[ |F_6| = 0 \]
\[ |F_7| = 1.35 \quad \angle F_7 = 20^\circ \]
\[ |F_8| = 2.53 \quad \angle F_8 = 100^\circ . \]

The plots of the magnitude and the phase spectra are shown in Figs. 5.1 and 5.2.

Fig. 5.1

Fig. 5.2
Question 5.3

Determine the DFT of the periodic sequence

\[ \{f_m\} = \{1, -1, 0, 0, 0, 0\} \]

and express the components in polar form. Determine directly the DFT of the shifted signal \( \{g_m\} \), where

\[ g_m = f_{m-1}. \]

Compare the result with that obtained using the shifting theorem.

Solution

To determine the DFT of the sequence \( \{f_m\} \) we use the equation

\[ F_n = \sum_{m=0}^{1} f_m w^{-mn}, \tag{5.6} \]

where

\[ w = e^{j\frac{\pi}{3}}. \tag{5.7} \]

Hence,

\[ F_n = 1 - e^{j\frac{\pi}{3}}. \]

The result is as follows

\[ F_0 = 0, \]
\[ F_1 = e^{j60^\circ}, \]
\[ F_2 = \sqrt{3}e^{j30^\circ}, \]
\[ F_3 = 2, \]
\[ F_4 = \sqrt{3}e^{-j30^\circ}, \]
\[ F_5 = e^{-j60^\circ}. \]

Now we determine the DFT of the shifted periodic signal

\[ \{g_m\} = \{0, 1, -1, 0, 0, 0\} \tag{5.8} \]

using the formula

\[ G_n = \sum_{m=1}^{3} g_m w^{-mn} = e^{-j\frac{\pi}{3}n} - e^{-j\frac{2\pi}{3}n}. \tag{5.9} \]

Setting \( n = 0, 1, ..., 5 \) we find

\[ G_0 = 0, \]
\[ G_1 = 1, \]
\[ G_2 = \sqrt{3}e^{-j\frac{\pi}{2}}, \]
\[ G_3 = 2e^{-j\pi}, \]
\[ G_4 = e^{-j\frac{\pi}{3}}, \]
\[ G_5 = e^{-j\frac{2\pi}{3}}. \]
\[ G_4 = \sqrt{3} e^{j\pi/2}, \]
\[ G_5 = 1. \]

To verify these results we apply the shifting theorem
\[ G_n = w^{-n} F_n = e^{-j\pi n/3} F_n. \] (5.10)

Hence, we obtain
\[
\begin{align*}
G_0 &= 0, \\
G_1 &= e^{-j60^\circ} e^{j60^\circ} = 1, \\
G_2 &= e^{-j120^\circ} \sqrt{3} e^{j30^\circ} = \sqrt{3} e^{-j\pi/2}, \\
G_3 &= e^{-j180^\circ} \cdot 2 = 2e^{-j\pi}, \\
G_4 &= e^{-j240^\circ} \sqrt{3} e^{-j30^\circ} = \sqrt{3} e^{j\pi/2}, \\
G_5 &= e^{-j300^\circ} e^{-j60^\circ} = e^{-j360^\circ} = 1.
\end{align*}
\]

**Question 5.4**

Determine the DFT of the periodic sequences:
\[
\{ f_m \} = \{ 2, 1, -1, 0 \}, \quad (5.11)
\]
\[
\{ g_m \} = \{ 1, 2, 1, -1 \}. \quad (5.12)
\]

Calculate the convolution of \( \{ f_m \} \) and \( \{ g_m \} \). Compare the obtained answer using the inverse DFT of \( F_n G_n \).

**Solution**

To calculate the convolution \( h_n = f_n * g_n = \sum_{m=0}^{N-1} f_m g_{n-m} \) we perform four steps: folding, translating, multiplying and adding. It is illustrated, for \( n = 1 \), in Figs. 5.3 – 5.7.
Fig. 5.4

Fig. 5.5

Fig. 5.6
Employing the plot shown in Fig. 5.7 yields

\[ h(t) = \sum_{n=0}^{3} f_n g_{1-m} = 6. \]

Similarly we find

\[ h_0 = 0, \quad h_2 = 3, \quad h_3 = -3 \]

leading to the convolution \( \{h_n\} = \{0, 6, 3, -3\} \).

To determine the DFT of the sequence \( \{f_n\} \) we use the equation

\[
F_n = \sum_{m=0}^{N-1} f_m e^{-j \frac{2\pi mn}{N}} = 2 + e^{-j \frac{\pi n}{2}} - e^{-j \frac{3\pi n}{2}}.
\]

Hence, we have

\[ F_0 = 2, \]
\[ F_1 = 3.162 e^{-j1.8^\circ}, \]
\[ F_2 = 0, \]
\[ F_3 = 3.162 e^{j1.8^\circ}. \]

Similarly we find the DFT of the sequence \( \{g_n\} \)

\[
G_n = \sum_{m=0}^{3} g_m e^{-j \frac{\pi mn}{2}} = 1 + 2e^{-j \frac{\pi n}{2}} + e^{-j \pi n} - e^{-j \frac{3\pi n}{2}}.
\]

Setting \( n = 0, 1, 2, 3 \) we obtain

\[ G_0 = 3, \quad G_1 = 3e^{-j \frac{\pi}{2}}, \quad G_2 = 1, \quad G_3 = 3e^{j \frac{\pi}{2}}. \]
Having \( \{F_n\} \) and \( \{G_n\} \) we form the product \( \{F_n G_n\} \) which is the DFT of the convolution \( \{h_n\} \)

\[
\{H_n\} = \{F_n G_n\} = \{6, 9.486e^{-j108.4^\circ}, 0, 9.486e^{j108.4^\circ}\}.
\]

We find the inverse DFT of \( \{H_n\} \)

\[
h_m = \frac{1}{N} \sum_{n=0}^{N-1} H_n w^{mn} = \frac{1}{4} \sum_{n=0}^{3} H_n e^{\frac{j\pi mn}{2}} = \frac{1}{4} \left( 6 + 9.486e^{-j108.4^\circ} e^{\frac{j\pi m}{2}} + 9.486e^{j108.4^\circ} e^{\frac{3j\pi m}{2}} \right).
\]

Hence, setting \( m = 0, 1, 2, 3 \) we have

\[
h_0 = 0, \ h_1 = 6, \ h_2 = 3, \ h_3 = -3,
\]

which confirms the result obtained directly.

**Question 5.5**

Determine the DFT of the periodic sequences:

\[
\{f_m\} = \{-1, -1, 0, 0\}, \quad (5.15)
\]

\[
\{g_m\} = \{1, 1, 1, 0\}. \quad (5.16)
\]

Using the inverse DFT find the convolution of the sequences.

**Solution**

The DFT of the sequence \( \{f_m\} \) is as follows

\[
F_n = \sum_{m=0}^{N-1} f_m w^{-mn} = -1 - e^{-j\frac{\pi n}{2}}. \quad (5.17)
\]

Hence, we find

\[
\{F_n\} = \{-2, -1+j, 0, -1-j\}.
\]

Similarly, we determine the DFT of \( \{g_m\} \)

\[
G_n = \sum_{m=1}^{N-1} g_m w^{-mn} = 1 + e^{-j\frac{\pi n}{2}} + e^{-j\pi m}. \quad (5.18)
\]

As a result, we obtain

\[
\{G_n\} = \{3, -j, 1, j\}.
\]

Next we compute \( H_n = F_n G_n \).
\{H_n\} = \{-6, 1+j, 0, 1-j\}

and find the sequence \{h_m\} using the inverse DFT

\[
h_m = \frac{1}{N} \sum_{n=0}^{N-1} H_n w^{mn} = \frac{1}{4} \sum_{n=0}^{3} H_n e^{\frac{j \pi mn}{2}} = \frac{1}{4} \left( -6 + (1+j)e^{\frac{j \pi}{2}} + (1-j)e^{\frac{3j \pi}{2}} \right),
\]

which leads to \{h_m\} = \{-1, -2, -2, -1\}.

**Question 5.6**

Determine the DFT of the periodic sequence

\{f_m\} = \{3, 2, 1, 0\}. \quad (5.19)

Compute the DFT of the shifted sequence \{g_m\}, where \(g_m = f_{m-2}\). Next, find the convolution of these sequences. Compare the answer with the result obtained using the convolution theorem.

**Solution**

To find the DFT of \{f_m\} we apply the formula

\[
F_n = \sum_{m=0}^{N-1} f_m w^{-mn} = 3 + 2e^{-\frac{j \pi m}{2}} + e^{-j \pi n}, \quad (5.20)
\]

which leads to the results:

\[
F_0 = 6, \quad F_2 = 2, \quad F_1 = 2 - j2 = 2\sqrt{2}e^{-j45^\circ}, \quad F_3 = 2 + j2 = 2\sqrt{2}e^{j45^\circ}.
\]

Sequences \{f_m\} and \{g_m\} are depicted in Figs. 5.8 and 5.9.
Hence,
\[
\{g_m\} = \{f_{m-2}\} = \{1, 0, 3, 2\}
\]
and its DFT can be determined using the formula
\[
G_n = \sum_{m=0}^{N-1} g_m w^{-mn} = 1 + 3e^{-j\pi n} + 2e^{-j\frac{3\pi}{2} n} .
\] (5.21)

As the result, we obtain
\[
G_0 = 6 ,
G_1 = -2 + j2 = 2\sqrt{2}e^{j35^\circ} ,
G_2 = 2 ,
G_3 = -2 - j2 = 2\sqrt{2}e^{-j35^\circ} .
\]

Now we compute the convolution
\[
h_n = f_n * g_n = \sum_{m=0}^{N-1} f_m g_{n-m} = \sum_{m=0}^{3} f_m g_{n-m} .
\] (5.22)

Using directly the equation (5.22) for \(n = 0, 1, 2, 3\) we find
\[
h_0 = 3 + 4 + 3 =10 ,
\]
\[
h_1 = 2 + 2 = 4 ,
\]
\[
h_2 = 9 + 1 = 10 ,
\]
\[
h_3 = 6 + 6 = 12 .
\]

To verify this answer we apply the convolution theorem. At first, we compute the sequence \(\{H_n\}\), where \(H_n = F_n G_n\). The result is
Next, we apply the inverse DFT

\[ h_m = \frac{1}{N} \sum_{n=0}^{N-1} H_n w^{mn} = \frac{1}{4} \left( 36 + 8 j e^{j \frac{2\pi}{N}} + 4 e^{j \frac{2\pi}{3}} - 8 j e^{j \frac{2\pi}{3}} \right), \]

finding \( h_0 = 10, \ h_1 = 4, \ h_2 = 10, \ h_3 = 12. \)

**Question 5.7**

Determine the DFT of the periodic sequence \( \{f_m\} = \{1, 2, 1, -2, 0\}. \) Compute the convolution of this sequence with itself. Compare the answer obtained using the convolution theorem.

**Solution**

To determine the DFT of \( \{f_m\} \) we apply the formula

\[ F_n = \sum_{m=0}^{N-1} f_m w^{-mn}, \quad n = 0, 1, ..., N-1, \]  

(5.23)

where \( N = 5 \) and

\[ w = e^{-j \frac{2\pi}{N}} = e^{-j \frac{2\pi}{5}}. \]  

(5.24)

Hence, it follows

\[ F_n = \sum_{m=0}^{3} f_m e^{-j \frac{2\pi}{5} mn}. \]  

(5.25)

The upper limit of the summation is 3 because \( f_4 = 0. \) Setting \( n = 0, 1, 2, 3, 4 \) we obtain

\[ F_0 = 2, \]

\[ F_1 = 2.427 - j3.665 = 4.396 e^{-j56.49^\circ}, \]

\[ F_2 = -0.927 + j1.678 = 1.917 e^{j18.93^\circ}, \]

\[ F_3 = -0.927 - j1.678 = 1.917 e^{-j18.93^\circ}, \]

\[ F_4 = 2.427 + j3.665 = 4.396 e^{j56.49^\circ}. \]

Next, we compute the convolution
Using equation (5.26) and the sequence representation shown in Fig. 5.10 we obtain

\[ h_n = f_n \ast f_n = \sum_{m=0}^{3} f_m f_{n-m}. \]  

(5.26)

On the basis of the convolution theorem we determine the DFT of the sequence \( \{h_n\} \) as the product of \( F_n \) and \( F_n \). Thus, \( H_n = F_n^2 \), \( (n = 0, 1, ..., 4) \), which leads to the sequence:

\[
\begin{align*}
H_0 &= 4, \\
H_1 &= 19.326e^{-j112.98^\circ}, \\
H_2 &= 3.674e^{-j122.15^\circ}, \\
H_3 &= 3.674e^{j122.15^\circ}, \\
H_4 &= 19.326e^{j112.98^\circ}.
\end{align*}
\]

To verify the convolution we apply the inverse DFT

\[
h_m = \frac{1}{N} \sum_{n=0}^{N-1} H_n w^{mn} = \frac{1}{5} \sum_{n=0}^{4} H_n e^{\frac{2\pi}{5} j mn}.
\]

(5.27)

Equation (5.27) leads to the following results:

\[
h_0 = -3, \quad h_1 = 8, \quad h_2 = 6, \quad h_3 = 0, \quad h_4 = -7.
\]
The answer is the same as computed directly.

**Question 5.8**

Applying the FFT algorithm find the DFT for the sequence \{f_m\} = \{1, 2, 3, 3\}.

**Solution**

We determine 2–point transform of the sequences \{f_0, f_2\} and \{f_1, f_3\}:

\[
G_0 = f_0 + f_2 , \\
G_1 = f_0 + w_2^{-1}f_2 ,
\]

where \( w_2 = e^{j\pi} \). The results are:

\[
G_0 = 1 + 3 = 4 , \quad H_0 = 2 + 3 = 5 , \\
G_1 = 1 + 3e^{-j\pi} = -2 , \quad H_1 = 2 + 3e^{-j\pi} = -1 .
\]

Next, we compute the DFT of the sequence \{f_m\} using the following equations:

\[
F_0 = G_0 + H_0 , \\
F_1 = G_1 + w_4^{-1}H_1 , \\
F_2 = G_0 - H_0 , \\
F_3 = G_1 - w_4^{-1}H_1 ,
\]

where \( w_4 = e^{j\pi/2} \). Substituting \( G_0, G_1, H_0, H_1 \), we obtain

\[
F_0 = 9 , \\
F_1 = -2 + j , \\
F_2 = -1 , \\
F_3 = -2 - j .
\]

**Question 5.9**

Repeat Question 5.8 for the sequence \{f_m\} = \{1+j4, 1+j3, -1-j, j\}.

**Solution**

At first, we find the 2–point transforms

\[
G_0 = f_0 + f_2 , \\
G_1 = f_0 + f_2w_2^{-1} ,
\]

and
\[ H_0 = f_1 + f_3, \]
\[ H_1 = f_1 + f_3 w_2^{-1}, \]

where \( w_2 = e^{j\pi} \). As a result, we obtain

\[ G_0 = j3, \]
\[ G_1 = 2 + j5, \]
\[ H_0 = 1 + j4, \]
\[ H_1 = 1 + j2. \]

Next, we substitute the above coefficients into the equations

\[ F_0 = G_0 + H_0, \]
\[ F_1 = G_1 + w_4^{-1} H_1, \]
\[ F_2 = G_0 - H_0, \]
\[ F_3 = G_1 - w_4^{-1} H_1, \]

where \( w_4 = e^{j\pi/4} \). Hence, we have

\[ F_0 = 1 + j7, \quad F_1 = 4 + j4, \quad F_2 = -1 - j, \quad F_3 = j6. \]
7. The Z-transform

Question 7.1

Find the Z-transform of the following discrete signals:

a) \( x_1(n) = 3\delta(n) + 7u(n) \),

b) \( x_2(n) = 18n3^{n-1} + 7n5^{n+1} \),

c) \( x_3(n) = 5n^27^{n-1} + 8(n + 1) \).

Solution

a) Since \( Z(\delta(n)) = 1 \) and \( Z(u(n)) = \frac{z}{z-1} \) we have

\[ X_1(z) = 3 + 7\frac{z}{z-1} . \]

b) We apply the differentiation rule

\[ Z(nx(n)) = -z \frac{dY(z)}{dz} \] (7.1)

to each term of \( x_2(n) \):

\[ Z(18n3^{n-1}) = Z(6n3^n) = 6Z(n3^n) = -6z \frac{d}{dz} \left( \frac{z}{z-3} \right) = \frac{18z}{(z-3)^2} , \]

\[ Z(7n5^{n+1}) = 35Z(n5^n) = 35(-z) \frac{d}{dz} \left( \frac{z}{z-5} \right) = \frac{175z^2}{(z-5)^2} . \]

Using above results and linearity property we obtain

\[ X_2(z) = 18\frac{z}{(z-3)^2} + 175\frac{z}{(z-5)^2} . \]

c) Signal \( x_3(n) \) can be rewritten as follows

\[ x_3(n) = \frac{5}{7}n(n7^n) + 8n + 8u(n) . \]

To find the Z-transform of the first term on the right hand side we apply the differentiation rule twice. First, we compute the Z-transform of signal \( n7^n \)

\[ Z(n7^n) = -z \frac{d}{dz} \left( \frac{z}{z-7} \right) = \frac{7z}{(z-7)^2} . \] (7.2)

Next we denote \( n7^n = w(n) \) and compute the Z-transform of the signal
using again the differentiation rule

\[ Z\left(\frac{5}{7}n(\theta^n)\right) = \frac{5}{7}Z(nw(n)) = \frac{5}{7}(-z) \frac{dW(z)}{dz} = -\frac{5}{7}z \frac{d}{dz} \left(\frac{7}{(z-7)^2}\right) = -5z \frac{-z^2 + 49}{(z-7)^4} = 5z \frac{(z-7)(z+7)}{(z-7)^4} = 5z\frac{z+7}{(z-7)^3}. \]

Since \( Z(n) = Z(nu(n)) = \frac{z}{(z-1)^2} \) then

\[ Z(8n) = 8 \frac{z}{(z-1)^2}. \]

The Z-transform of the last term is

\[ Z(8u(n)) = 8 \frac{z}{z-1}. \]

Using the above results and the linearity property we obtain

\[ X_3(z) = 5\frac{z(z+7)}{(z-7)^3} + 8 \frac{z}{(z-1)^2} + 8 \frac{z}{z-1}. \]

**Question 7.2**

Compute the Z-transform of the signal \( x(n) = \cosh(\omega_nT) \).

**Solution**

We express \( \cosh(\omega_nT) \) in the form

\[ \cosh(\omega_nT) = \frac{1}{2}(e^{\omega_nT} + e^{-\omega_nT}) \quad (7.3) \]

and apply the formula

\[ Z(e^{-\omega_nT}) = Z(\left(e^{\omega_nT}\right)^n) = \frac{z}{z-e^{-\omega_nT}}. \quad (7.4) \]

As a result, we obtain

\[ X(z) = Z(\cosh(\omega_nT)) = \frac{1}{2}Z(e^{\omega_nT}) + \frac{1}{2}Z(e^{-\omega_nT}) = \frac{1}{2}\frac{z}{2z-e^{\omega_nT}} + \frac{1}{2}\frac{z}{2z-e^{-\omega_nT}} = \frac{1}{2}z \frac{2z-e^{-\omega_nT}-e^{\omega_nT}}{(z-e^{\omega_nT})(z-e^{-\omega_nT})} = \frac{z(z - \cosh(\omega_nT))}{z^2 - 2z\cosh(\omega_nT) + 1}. \]

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Question 7.3

Using appropriate theorems find the initial and the final values of signals having the following Z-transforms:

a) \( X_1(z) = \frac{z^2}{z^2 - 0.16} \),
b) \( X_2(z) = \frac{z(z + 1)}{(z - 1)(z - 0.5)(z + 0.2)} \).

Solution

a) Since the function \( \frac{z - 1}{z} X_1(z) = \frac{(z - 1)z}{(z - 0.4)(z + 0.4)} \) has singularities inside the unit circle only, the equation

\[ x_1(\infty) = \lim_{z \to 1} \frac{(z - 1)z}{(z - 0.4)(z + 0.4)} = 0 \]

holds. The result can be easily predicted because no singularity of \( X_1(z) \) occurs at \( z = 1 \).

b) The function \( \frac{z - 1}{z} X_2(z) = \frac{z + 1}{(z - 0.5)(z + 0.2)} \) has the singularities inside the unit circle only, hence, the final-value theorem gives

\[ x_2(\infty) = \lim_{z \to 1} \frac{z + 1}{(z - 0.5)(z + 0.2)} = \frac{10}{3}. \]

The final value is unequal to zero because \( X_2(z) \) has a singularity at \( z = 1 \).

The initial value are as follows

\[ x_1(0) = \lim_{z \to \infty} X_1(z) = 1, \]
\[ x_2(0) = \lim_{z \to \infty} X_2(z) = 0. \]

Question 7.4

Find the inverse Z-transform of the following functions:

a) \( F_1(z) = \frac{z + 1}{(z - 1)(z - 2)} \),
Solution

a) We arrange the partial fraction expansion of \( \frac{F_1(z)}{z} \):

\[
F_1(z) = \frac{z + 1}{z(z - 1)(z - 2)} = \frac{k_1}{z - 1} + \frac{k_2}{z - 2} + \frac{k_3}{z - 1}.
\]

(7.5)

where

\[
k_1 = \lim_{z \to 0} z \frac{F_1(z)}{z} = \frac{1}{2},
\]

\[
k_2 = \lim_{z \to 1} (z - 1) \frac{F_1(z)}{z} = -2,
\]

\[
k_3 = \lim_{z \to 2} (z - 2) \frac{F_1(z)}{z} = \frac{3}{2}.
\]

Thus,

\[
F_1(z) = \frac{1}{2} z - 2 - \frac{z}{z - 1} + \frac{3}{2} \frac{z}{z - 2},
\]

(7.6)

and the inverse Z-transform is

\[
f_1(n) = \frac{1}{2} \delta(n) - 2u(n) + \frac{3}{2} 2^n.
\]

(7.7)

Setting \( n = 0, 1, 2, 3, 4, \ldots \) we have \( \{f_1(n)\} = \{0, 1, 4, 10, 22, \ldots\} \).

Alternatively, the inversion can be determined performing the long division of the numerator and denominator polynomials

\[
(z + 1) \left( z^2 - 3z + 2 \right) = z^{-1} + 4z^{-2} + 10z^{-3} + 22z^{-4} + \ldots
\]

\[
-3z + 3 - 2z^{-1}
\]

\[
\quad 4 - 2z^{-1}
\]

\[
-4 + 12z^{-1} - 8z^{-2}
\]

\[
\quad 10z^{-1} - 8z^{-2}
\]

\[
-10z^{-1} + 30z^{-2} - 20z^{-3}
\]

\[
\quad 22z^{-2} - 20z^{-3}
\]

The coefficients of the power series

\[
0z^0 + z^{-1} + 4z^{-2} + 10z^{-3} + 22z^{-4} + \ldots
\]

form the signal \( \{f_1(n)\} = \{0, 1, 4, 10, 22, \ldots\} \) which is the same as obtained using the partial fraction expansion approach.
b) We form function \( \frac{F_2(z)}{z} \) and express it in terms of partial fraction expansion

\[
\frac{F_2(z)}{z} = \frac{1}{z(z-0.2)(z-1)^2} = k_1 + \frac{k_2}{z-0.2} + \frac{k_{31}}{(z-1)^2} + \frac{k_{32}}{z-1},
\]

where

\[
k_1 = \lim_{z \to 0} z \frac{F_2(z)}{z} = \lim_{z \to 0} \frac{1}{z(z-0.2)(z-1)^2} = -5,
\]

\[
k_2 = \lim_{z \to 0.2} \frac{z}{z-0.2} \frac{F_2(z)}{z} = \lim_{z \to 0.2} \frac{1}{z(z-0.2)^2} = 7.8125,
\]

\[
k_{31} = \lim_{z \to 1} (z-1)^2 \frac{F_2(z)}{z} = \frac{1}{z-0.2} = 1.25,
\]

\[
k_{32} = \lim_{z \to 1} \frac{z}{z-0.2} \frac{F_2(z)}{z} = \lim_{z \to 1} \frac{1}{z(z-0.2)^2} = -2.8125.
\]

Thus,

\[
F_2(z) = -5 + 7.8125 \frac{z}{z-0.2} + 1.25 \frac{z}{(z-1)^2} - 2.8125 \frac{z}{z-1}
\]

(7.9)

and the inverse Z-transform is

\[
f_2(n) = -5\delta(n) + 7.8125(0.2)^n + 1.25nu(n) - 2.8125u(n).
\]

(7.10)

c) The rational function \( F_3(z) \) has two poles which form a complex conjugate pair

\[
\begin{align*}
p_1 &= \frac{\sqrt{3}}{2} + j\frac{1}{2} = e^{j30^\circ}, \\
p_2 &= \frac{\sqrt{3}}{2} - j\frac{1}{2} = p_1^*.
\end{align*}
\]

The partial fraction expansion of this function is

\[
\frac{F_3(z)}{z} = \frac{2z + 1}{z \left(z - \left(\frac{\sqrt{3}}{2} + j\frac{1}{2}\right)\right) \left(z - \left(\frac{\sqrt{3}}{2} - j\frac{1}{2}\right)\right)} = \frac{k_1}{z} + \frac{k_2}{z - p_1} + \frac{k_2^*}{z - p_1^*},
\]

where

\[
k_1 = \lim_{z \to 0} z \frac{F_3(z)}{z} = 1
\]
\[ k_2 = \lim_{z \to \left( \frac{\sqrt{3}}{2} + j \frac{1}{2} \right)} \left( z - \left( \frac{\sqrt{3}}{2} + j \frac{1}{2} \right) \right) \frac{F_3(z)}{z} = \frac{2 \left( \frac{\sqrt{3}}{2} + j \frac{1}{2} \right) + 1}{\left( \frac{\sqrt{3}}{2} + j \frac{1}{2} \right) j} = \frac{1}{2} (-1 + j \sqrt{3}) \]

\[ = 2 \left( \frac{\sqrt{3} + 1}{4} \right) (-1 - j \sqrt{3}) = \frac{1}{2} \left( -\sqrt{3} - 1 + \sqrt{3} + j(-1 - 3 + \sqrt{3}) \right) = \frac{-1 - j \frac{4 + \sqrt{3}}{2}}{2} = -0.5 - j 2.866 = 2.91 e^{j 260.1^\circ}. \]

Thus,

\[ F_3(z) = 1 + 2.91 e^{j 260.1^\circ} \frac{z}{z - e^{j 30^\circ}} + 2.91 e^{-j 260.1^\circ} \frac{z}{z - e^{-j 30^\circ}} \]

holds and the inverse Z-transform is

\[ f_3(n) = \delta(n) + 2.91 e^{j 260.1^\circ} e^{j 30^\circ} n + 2.91 e^{-j 260.1^\circ} e^{-j 30^\circ} n = \delta(n) + 5.82 \cos \left( 30^\circ n + 260.1^\circ \right). \tag{7.11} \]

**Question 7.5**

Evaluate the convolution of \( x_1(n) \) and \( x_2(n) \), where

\[ \left\{ x_1(n) \right\} = \{3, 2, 1, -1, -1\}, \]

\[ \left\{ x_2(n) \right\} = \{-1, 1, 0, 0, 1\}, \]

and determine directly the Z-transform of the convolution \( x(n) = x_1(n) * x_2(n) \). Verify the result using the convolution theorem.

**Solution**

To determine the convolution we use the graphical approach illustrated, at \( n = 1 \), in Fig.7.1
On the basis of Fig. 7.1 and the formula

\[ x(1) = \sum_{k=0}^{4} x_1(k)x_2(1-k) \]

we obtain \( x(1) = 1 \).

Repeating the procedure for \( n = 0, 2, 3, 4, \ldots \) we find:

\[
\begin{align*}
    x(0) &= -3, \\
    x(2) &= 1, \\
    x(3) &= 2, \\
    x(4) &= 3, \\
    x(5) &= 1, \\
    x(6) &= 1, \\
    x(7) &= 2, \\
    x(8) &= 3, \\
    x(n) &= 0 \text{ for } n > 8.
\end{align*}
\]

Hence, the \( Z \)-transform of the convolution is

\[ X(z) = -3 + z^{-1} + z^{-2} + 2z^{-3} + 3z^{-4} + z^{-5} + z^{-6} - z^{-7} - z^{-8}. \]

To verify this result we apply the convolution theorem

\[ X(z) = X_1(z)X_2(z), \quad (7.12) \]

where

\[
\begin{align*}
    X_1(z) &= 3 + 2z^{-1} + z^{-2} - z^{-3} - z^{-4}, \\
    X_2(z) &= -1 + z^{-1} + z^{-4}.
\end{align*}
\]

As a result, we obtain

\[ X(z) = \left(3 + 2z^{-1} + z^{-2} - z^{-3} - z^{-4}\right)\left(-1 + z^{-1} + z^{-4}\right) = -3 + z^{-1} + z^{-2} + 2z^{-3} + 3z^{-4} + z^{-5} + z^{-6} - z^{-7} - z^{-8}. \]

**Question 7.6**

A system is characterised by the difference equation
\[ y(n) - y(n-1) - 0.5y(n-2) = x(n) - 0.6x(n-1). \] (7.13)

Determine the system transfer function and the unit pulse response.

**Solution**

We compute the \( Z \)-transform of this equation assuming zero initial conditions \( y(-1) = 0 \), \( y(-2) = 0 \), \( x(-1) = 0 \)

\[ Y(z) - z^{-1}Y(z) - 0.5z^{-2}Y(z) = X(z) - 0.6z^{-1}X(z). \]

Hence, it follows

\[ H(z) = \frac{Y(z)}{X(z)} = \frac{1 - 0.6z^{-1}}{1 - z^{-1} - 0.5z^{-2}} = \frac{z(z - 0.6)}{z^2 - z - 0.5}. \] (7.14)

To determine the unit pulse response we compute the inverse \( Z \)-transform of \( H(z) \) using the standard approach as shown below

\[ H(z) = \frac{z - 0.6}{z^2 - z - 0.5} = \frac{z - 0.6}{(z + 0.366)(z - 1.366)} = \frac{k_1}{z + 0.366} + \frac{k_2}{z - 1.366}, \]

where \( k_1 = 0.558 \) and \( k_2 = 0.442 \).

Thus,

\[ H(z) = 0.558 \frac{z}{z + 0.366} + 0.442 \frac{z}{z - 1.366} \]

and the unit pulse response is

\[ h(n) = Z^{-1}(H(z)) = 0.558(-0.366)^n + 0.422 \cdot 1.366^n. \]

**Question 7.7**

Determine the unit pulse response of a system characterised by the transfer function

\[ H(z) = \frac{2z}{(z + 2)(z - 1)^2(z^2 - 2z + 2)}. \] (7.15)

**Solution**

The unit pulse response is the inverse \( Z \)-transform of the transfer function. To find the inversion we arrange the partial fraction expansion of \( \frac{H(z)}{z} \):

\[
\frac{H(z)}{z} = \frac{2}{(z + 2)(z - 1)^2(z - (1 - j))(z - (1 + j))} = \\
= \frac{k_1}{z + 2} + \frac{k_{21}}{(z - 1)^2} + \frac{k_{22}}{z - 1} + \frac{k_3}{z - (1 - j)} + \frac{k_4}{z - (1 + j)},
\]

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Thus, \(H(z)\) can be presented in the form

\[
H(z) = \frac{1}{45} \frac{z}{z+2} + \frac{2}{3} \frac{z}{(z-1)^2} - \frac{2}{9} \frac{z}{z-1} + \sqrt{0.1} e^{-j1.6} \frac{1}{z - (1-j)} + \sqrt{0.1} e^{j1.6} \frac{1}{z - (1+j)}
\]

and the inverse Z–transform of \(H(z)\) is

\[
h(n) = \frac{1}{45} (-2)^n + \frac{2}{3} n u(n) - \frac{2}{9} u(n) + \sqrt{0.1} e^{-j1.6} \left( \sqrt{2} e^{-j45} \right)^n + \sqrt{0.1} e^{j1.6} \left( \sqrt{2} e^{j45} \right)^n = \\
= \frac{1}{45} (-2)^n + \frac{2}{3} n u(n) - \frac{2}{9} u(n) + 2 \sqrt{0.1} \left( \sqrt{2} \right)^n \cos(n45^\circ + 71.6^\circ).
\]

**Question 7.8**

Consider the discrete system represented by the diagram depicted in Fig. 7.2. Find the transfer function describing the system. Determine the response of the system to the unit step function. Write the difference equation describing the system.
Solution

On the basis of the system diagram we write

\[ W(z) = 0.5X(z) + 0.25z^{-1}Y(z) \]  \hspace{1cm} (7.16)

\[ Y(z) = 0.4X(z) + z^{-1}W(z). \]  \hspace{1cm} (7.17)

We substitute \( W(z) \) given by the equation (7.16) into the equation (7.17)

\[ Y(z) = 0.4X(z) + 0.5z^{-1}X(z) + 0.25z^{-2}Y(z). \]  \hspace{1cm} (7.18)

Hence, we have

\[ H(z) = \frac{Y(z)}{X(z)} = \frac{0.4 + 0.5z^{-1}}{1 - 0.25z^{-2}} = \frac{z(0.4z + 0.5)}{z^2 - 0.25} = \frac{z(0.4z + 0.5)}{(z - 0.5)(z + 0.5)}. \]

Since the \( Z \)-transform of the discrete unit step function is

\[ Z[u(n)] = \frac{z}{z - 1}, \]  \hspace{1cm} (7.19)

the \( Z \)-transform of the unit step response of the system is

\[ Y(z) = H(z) \frac{z}{z - 1} = \frac{z^2(0.4z + 0.5)}{(z - 1)(z - 0.5)(z + 0.5)}. \]  \hspace{1cm} (7.20)

To find the unit step response we calculate the inverse \( Z \)-transform of \( Y(z) \) using the standard approach

\[ Y(z) = \frac{z(0.4z + 0.5)}{(z - 1)(z - 0.5)(z + 0.5)} = 1.2 \frac{z}{z - 1} - 0.7 \frac{z}{z - 0.5} - 0.3 \frac{z}{z + 0.5}, \]

hence,

\[ Y(z) = 1.2 \frac{z}{z - 1} - 0.7 \frac{z}{z - 0.5} - 0.3 \frac{z}{z + 0.5}. \]  \hspace{1cm} (7.21)

The expression on the right hand side of (7.21) has the form which enables us to determine the unit step response directly

\[ y(n) = 1.2u(n) - 0.7(0.5)^n - 0.3(-0.5)^n. \]  \hspace{1cm} (7.22)

To write the difference equation describing the system we use the equation

\[ \frac{Y(z)}{X(z)} = \frac{0.4 + 0.5z^{-1}}{1 - 0.25z^{-2}} \]  \hspace{1cm} (7.23)

and rearrange it as follows
\[ Y(z) - 0.25z^{-2}Y(z) = 0.4X(z) + 0.5z^{-1}X(z). \] (7.24)

The inverse Z–transform of both sides of this equation leads to the difference equation

\[ y(n) - 0.25y(n - 2) = 0.4x(n) + 0.5x(n - 1). \] (7.25)

**Question 7.9**

Using the Z–transform method solve the following difference equations:

a) \( x(n) = 3x(n - 1) + u(n) \), \( x(-1) = -1 \),

b) \( x(n) = 2x(n - 1) - x(n - 2) \), \( x(-1) = 1, \ x(-2) = -3 \),

c) \( x(n) = x(n - 1) + e^{-n} \), \( x(-1) = 3 \),

d) \( x(n) = 2x(n - 1) + x(n - 2) - \frac{1}{2}x(n - 3) \), \( x(-1) = 0, \ x(-2) = 0, \ x(-3) = -2 \).

**Solution**

a) Using the Z–transform and the shifting property we have

\[ X(z) = 3\left[ x(-1) + z^{-1}X(z) \right] + \frac{z}{z - 1}. \]

We set the initial condition \( x(-1) = -1 \) and solve the above for \( X(z) \)

\[ X(z) = \frac{1}{1 - 3z^{-1}} \left[ -3 + \frac{z}{z - 1} \right] = -\frac{3z}{z - 3} + \frac{z^2}{(z - 3)(z - 1)}. \]

The inverse Z–transform of \( X(z) \) gives

\[ x(n) = -3 \cdot 3^n + x_1(n), \tag{7.26} \]

where

\[ x_1(n) = Z^{-1}\left[ \frac{z^2}{(z - 3)(z - 1)} \right] = Z^{-1}(X_1(z)). \]

To compute \( x_2(n) \) we express \( \frac{X_1(z)}{z} \) via a partial fraction expansion

\[ \frac{X_1(z)}{z} = \frac{3}{2} \frac{1}{z - 3} - \frac{1}{2} \frac{1}{z - 1} \]

and determine \( X_1(z) \)

\[ X_1(z) = \frac{3}{2} \frac{z}{z - 3} - \frac{1}{2} \frac{z}{z - 1}. \]

The inverse Z–transform of \( X_1(z) \) is
\[ x_i(n) = \frac{3}{2} 3^n_u(n). \]

Setting \( x_i(n) \) into (7.26) yields

\[ x(n) = -3 \cdot 3^n + \frac{3}{2} 3^n_u(n) = -\frac{3}{2} 3^n - \frac{1}{2} u(n). \quad (7.27) \]

b) Application of the Z–transform to the difference equation gives

\[ X(z) = 2[z^{-1}X(z) + x(-1)] - [z^{-2}X(z) + z^{-1}x(-1) + x(-2)]. \quad (7.28) \]

We substitute the initial conditions and solve the equation for \( X(z) \)

\[ X(z) = \frac{5 - z^{-1}}{1 - 2z^{-1} + z^{-2}} = \frac{5z^2 - z}{z^2 - 2z + 1} = \frac{5z^2 - z}{(z - 1)^2}. \]

Next we consider the function \( \frac{X(z)}{z} \) and express it via a partial fraction expansion

\[ \frac{X(z)}{z} = \frac{5z - 1}{(z - 1)^2} = \frac{4}{(z - 1)^2} + \frac{5}{z - 1}. \]

Hence,

\[ X(z) = 4 \frac{z}{(z - 1)^2} + 5 \frac{z}{z - 1} \]

holds and the signal \( x(n) \) is

\[ x(n) = 4nu(n) + 5u(n). \quad (7.29) \]

c) We perform the Z–transform to the difference equation

\[ X(z) = z^{-1}X(z) + x(-1) + \frac{z}{z - e^{-1}} \quad (7.30) \]

and solve above equation for \( X(z) \)

\[ X(z) = \frac{x(-1)}{1 - z^{-1}} + \frac{z}{(1 - z^{-1})(z - e^{-1})} = \frac{3z}{z - 1} + \frac{z^2}{(z - 1)(z - e^{-1})}. \]

The inverse Z–transform of \( X(z) \) is

\[ x(n) = 3u(n) + x_1(n), \quad (7.31) \]

where \( x_1(n) \) is the inverse Z–transform of \( X_1(z) \). To determine \( x_1(n) \) via inversion of the rational function \( X_1(z) \) we use the standard approach as follows:
\[
X_1(z) = \frac{z}{z(z-1)(z-e^{-1})} = \frac{k_1}{z-1} + \frac{k_2}{z-e^{-1}},
\]
where \( k_1 = \frac{1}{1-e^{-1}}, \ k_2 = \frac{e^{-1}}{e^{-1}-1} \). Therefore, it holds

\[
X_1(z) = \frac{1}{1-e^{-1}} \frac{z}{z-1} + \frac{e^{-1}}{e^{-1}-1} \frac{z}{z-e^{-1}}.
\]

Thus, the inverse Z–transform of \( X_1(z) \) is

\[
x_1(n) = \frac{1}{1-e^{-1}} u(n) + \frac{e^{-1}}{e^{-1}-1} e^{-n}.
\]  \hspace{1cm} (7.32)

Plugging (7.32) into (7.31) yields

\[
x(n) = \frac{4-3e^{-1}}{1-e^{-1}} u(n) + \frac{e^{-1}}{e^{-1}-1} e^{-n} = \frac{4e-3}{e-1} u(n) + \frac{1}{1-e} e^{-n}.
\]  \hspace{1cm} (7.33)

d) We take the Z–transform of the difference equation

\[
X(z) = 2(z^{-1}X(z) + x(-1)) + z^{-2}X(z) + z^{-1}x(-1) + x(-2) + \frac{1}{2}(z^{-3}X(z) + z^{-2}x(-1) + z^{-1}(-2) + x(-3)) = 2z^{-1}X(z) + z^{-2}X(z) - \frac{1}{2}z^{-3}X(z) + 1
\]

and solve the transformed equation for \( X(z) \)

\[
X(z) = \frac{1}{1-2z^{-1}-z^{-2} + \frac{1}{2}z^{-3}} = \frac{z^3}{z^3 - 2z^2 - z + 0.5}.
\]

Next we consider \( \frac{X(z)}{z} \)

\[
\frac{X(z)}{z} = \frac{z^2}{z^3 - 2z^2 - z + 0.5} = \left( \frac{z}{z + 1.514} \right) \left( \frac{z}{z - 0.428} \right) \left( \frac{z}{z - 3.086} \right)
\]

and arrange the partial fraction expansion of this function

\[
\frac{X(z)}{z} = 0.256 \left( \frac{1}{z + 1.514} \right) - 0.355 \left( \frac{1}{z - 0.428} \right) + 0.779 \left( \frac{1}{z - 3.086} \right).
\]

Hence, we have

\[
X(z) = 0.256 \left( \frac{z}{z + 1.514} \right) - 0.355 \left( \frac{z}{z - 0.428} \right) + 0.779 \left( \frac{z}{z - 3.086} \right)
\]  \hspace{1cm} (7.34)

and the inverse Z–transform of this function can be computed term by term giving
\[ x(n) = 0.256(-1.514)^n - 0.355 \cdot 0.428^n + 0.779 \cdot 3.086^n. \]  

**Question 7.10**

The block diagram in Fig.7.3 represents a discrete–time data–transmission path. Find the output signal when the input signal is the unit pulse and the initial condition is \( w(-1) = 0.5 \).

![Fig. 7.3]

**Solution**

On the basis of the block diagram we write the following equations describing the system

\[
y(n) = w(n) + 0.2x(n),
\]

\[
w(n) = x(n) + 0.5w(n-1).
\]

Next, we take the \( Z \)–transform of these equations

\[
Y(z) = W(z) + 0.2X(z),
\]

\[
W(z) = X(z) + 0.5\left(z^{-1}W(z) + w(-1)\right) = X(z) + 0.5z^{-1}W(z) + 0.25.
\]

We solve the second equation for \( W(z) \)

\[
W(z) = \frac{X(z) + 0.25}{1 - 0.5z^{-1}}
\]

and substitute into the equation (7.38)

\[
Y(z) = \frac{X(z) + 0.25}{1 - 0.5z^{-1}} + 0.2X(z).
\]

Since \( x(n) \) is the unit pulse we set \( X(z) = 1 \)

\[
Y(z) = 1.25\frac{z}{z - 0.5} + 0.2.
\]

Thus, the inverse \( Z \)–transform is

\[
x(n) = 1.25(0.5)^n + 0.2\delta(n).
\]
Question 7.11

Implement the difference equation

\[ x(n) = x(n-1) + 2x(n-2) - x(n-3) + 5u(n) - 2u(n-1). \]  \tag{7.43}  

Solution

Equation (7.43) can be rewritten in the form

\[ x(n) = w(n) + x(n-1) + 2x(n-2) - x(n-3), \]

where

\[ w(n) = 5u(n) - 2u(n-1). \]

The representation of \( w(n) \) is shown in Fig. 7.4.

![Fig. 7.4](image)

To obtain \( x(n) \) we combine the signals \( w(n) \) with delayed values of the output signal as shown in Fig. 7.5.

![Fig. 7.5](image)
Next, we combine the representations of Fig. 7.4 and Fig. 7.5 as depicted in Fig. 7.6.

\[
\begin{align*}
    u(n) & \quad 5 \quad w(n) \quad x(n) \\
    z^{-1} & \quad -2 \\
    \end{align*}
\]