

12. Signal Energy and Power

12.1. Energy and power for continuous-time signals

The terms signal energy and signal power are used to characterize a signal. They are not actually measures of energy and power. The definition of signal energy and power refers to any signal $x(t)$, including signals that take on complex values.

Definition 1

The signal energy in the signal $x(t)$ is

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt . \quad (12.1)$$

The signal power in the signal $x(t)$ is

$$P = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt . \quad (12.2)$$

If $0 < E < \infty$, then the signal $x(t)$ is called an energy signal. However, there are signals where this condition is not satisfied. For such signals we consider the power. If $0 < P < \infty$, then the signal is called a power signal. Note that the power for an energy signal is zero ($P = 0$) and that the energy for a power signal is infinite ($E = \infty$). Some signals are neither energy nor power signals.

Let us consider a periodic signal $x(t)$ with period T_0 . The signal energy in one period is

$$E_1 = \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} |x(t)|^2 dt$$

and energy in n periods is

$$E_n = nE_1 = n \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} |x(t)|^2 dt .$$

The power of this signal over all periods is given by

$$P = \lim_{n \rightarrow \infty} \frac{nE_1}{nT_0} = \frac{1}{T_0} E_1 = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} |x(t)|^2 dt . \quad (12.3)$$

If the signal energy over one period is larger than zero but finite, then the total energy is infinite and the signal power is finite. Therefore, the signal is a power signal.

If the signal energy in one period is infinite, then both the power and the total energy are infinite. Consequently, the signal is neither an energy signal nor a power signal.

Consider a current signal $i(t)$ flowing through a transmission line represented by resistance R . The energy loss in the line is

$$E_R = \int_{-\infty}^{\infty} R(i(t))^2 dt = RE_i$$

where E_i is the signal energy in the signal $i(t)$. If the current $i(t)$ is periodic with period T_0 , the average power loss in the line is given by

$$P_{av} = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} R(i(t))^2 dt = RP_i$$

where P_i is the power of the periodic current signal $i(t)$.

In the special case in which $R = 1\Omega$ the energy loss in the line is the same as the signal energy in $i(t)$ and the power loss is the same as the power of the periodic signal $i(t)$.

Example 12.1

Let us consider a signal

$$x(t) = e^{-t} u(t).$$

The energy of this signal is

$$E = \int_{-\infty}^{\infty} |e^{-t} u(t)|^2 dt = \int_0^{\infty} e^{-2t} dt = -\frac{1}{2} e^{-2t} \Big|_0^{\infty} = \frac{1}{2}.$$

The signal $x(t)$ is an energy signal. Since E is finite the signal power $P = 0$.

Example 12.2

Let us consider a complex signal

$$x(t) = Ae^{j\omega_0 t}.$$

Signal $x(t)$ is periodic with period $T_0 = \frac{2\pi}{\omega_0}$; hence, it cannot be an energy signal.

To compute the signal power we use (12.3)

$$P = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} |Ae^{j\omega_0 t}|^2 dt = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} A^2 dt = A^2.$$

Since P is finite, $x(t)$ is a power signal and its energy is infinite.

The signal energy can be also expressed in the frequency domain, as shown below.

Let us take into account equation (12.1) and rearrange it as follows

$$\begin{aligned}
E &= \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t)x^*(t)dt = \int_{-\infty}^{\infty} x(t)\mathcal{F}^{-1}(X^*(-j\omega))dt = \\
&= \int_{-\infty}^{\infty} x(t)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(-j\omega)e^{j\omega t}d\omega\right)dt = \int_{-\infty}^{\infty} x(t)\left(-\frac{1}{2\pi}\int_{\infty}^{-\infty} X^*(j\omega)e^{-j\omega t}d\omega\right)dt = \\
&= \int_{-\infty}^{\infty} x(t)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)e^{-j\omega t}d\omega\right)dt
\end{aligned}$$

where the conjugate property (9.26) has been used.
Now we reverse the order of integration

$$\begin{aligned}
E &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)\left(\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt\right)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)X(j\omega)d\omega = \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega.
\end{aligned}$$

Thus, the equation

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega \quad (12.4)$$

holds. Expression (12.4) is known as Parseval's relation.

Parseval's relation states that the total energy may be determined either by integrating $|x(t)|^2$ over all time or by integrating $\frac{1}{2\pi}|X(j\omega)|^2$ over all frequencies. Therefore $|X(j\omega)|^2$ is interpreted as an energy spectral density of signal $x(t)$.

12.2. Energy and power for discrete-time signals

The definition of signal energy and power for discrete signals parallel similar definitions for continuous signals.

Definition 2

The signal energy in the discrete-time signal $x(n)$ is

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2. \quad (12.5)$$

The signal power in the signal $x(n)$ is

$$P = \lim_{N \rightarrow \infty} \left(\frac{1}{2N+1} \sum_{n=-N}^N |x(n)|^2 \right). \quad (12.6)$$

A discrete-time energy signal is defined as one for which $0 < E < \infty$ and a discrete-time power signal is defined as one for which $0 < P < \infty$. It is possible for a discrete-time signal to be neither an energy signal nor a power signal.

Example 12.3

Compute the signal energy and signal power for the discrete-time signal

$$x(n) = \left(\frac{1}{4}\right)^n u(n).$$

We apply relationship (12.5)

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=0}^{\infty} \left| \left(\frac{1}{4}\right)^n \right|^2 = \sum_{n=0}^{\infty} \left(\frac{1}{16}\right)^n.$$

The expression on the right hand side is a geometric series; hence, we have

$$E = \frac{1}{1 - \frac{1}{16}} = \frac{16}{15}.$$

Since $0 < E < \infty$, signal $x(n)$ is an energy signal, consequently, $P = 0$.

Example 12.4

Compute the signal power and signal energy for the discrete-time signal

$$x(n) = e^{10jn} u(n).$$

We apply equation (12.6)

$$P = \lim_{N \rightarrow \infty} \left(\frac{1}{2N+1} \sum_{n=-N}^N |x(n)|^2 \right) = \lim_{N \rightarrow \infty} \left(\frac{1}{2N+1} \sum_{n=0}^N 1^2 \right) = \lim_{N \rightarrow \infty} \frac{N+1}{2N+1} = \frac{1}{2}.$$

Since $0 < P < \infty$, the signal $x(n)$ is a power signal and its energy is infinite.

Let $x(n)$ be a discrete-time signal and $X(e^{j\tilde{\omega}})$ its DTFT. Then, the equation

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\tilde{\omega}})|^2 d\tilde{\omega} \quad (12.7)$$

holds.

Expression (12.7) is known as Parseval's relation for a discrete-time signal. This result is similar to (12.4) and can be derived in a similar manner. Parseval's relation states that the discrete-time signal energy can be determined either using

(12.5) or integrating $\frac{1}{2\pi} |X(e^{j\tilde{\omega}})|^2$ over a full 2π interval. Similarly as in the continuous-time case, $|X(e^{j\tilde{\omega}})|^2$ is called an energy spectral density of the signal $x(n)$.