Stability in electronic circuits

\[ \frac{dx}{dt} = f(x, t) \]

**Definition 1**
Consider a solution \( x^0(t) \) generated by an initial condition \( x^0(0) \) and a neighboring solution \( x(t) \). The solution \( x^0(t) \) is stable if for any \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that if

\[ |x(0) - x^0(0)| < \delta \]

then

\[ |x(t) - x^0(t)| < \epsilon \quad \text{for all} \quad t \geq 0 \]

If \( x^0(t) \) is not stable, it is said to be unstable.

**Definition 2**
\( x^0 \) is said to be asymptotically stable if it is stable, and if

\[ \lim_{t \to \infty} |x(t) - x^0(t)| = 0 \]

Fig. 1
Autonomous equation
\[
\frac{dx}{dt} = f(x) \quad x^0 - \text{the equilibrium point, } f(x^0) = 0
\]
\[
x = x^0 + y
\]
\[
\frac{dy}{dt} = f(y + x^0)
\]
\[
\frac{dy}{dt} = g(y)
\]
\[
g(y) = f(y + x^0)
\]
\[
g(0) = f(x^0) = 0
\]

0 is the equilibrium point of
\[
\frac{dy}{dt} = g(y)
\]
\[
\frac{dx}{dt} = f(x, t)
\]
\[
x = [x_1 \cdots x_n]^T
\]
\[
\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}
\]

Stability of the circuit as a whole (global stability)

**Definition 3**
A circuit is said to be completely stable if for any solution \( x(t) \)
\[
\lim_{t \to \infty} x(t) = 0
\]

**Definition 4**
A circuit is said to be Lagrange stable if all solutions remain bounded as \( t \to \infty \); that is, given any initial condition \( x(0) \), there exists an \( M \) (a function of \( x(0) \)), such that \( \|x(t)\| < M \) for all \( t \geq 0 \). If a circuit is not Lagrange stable, it will be called unstable.

Local stability, Lyapunov’s first method
\[
\frac{dx}{dt} = f(x)
\]
\[
x = x^0 + y
\]
\[
\frac{dy}{dt} = f(x^0 + y)
\]

\[
f(x^0 + y) = f(x^0) + \frac{df}{dx}(x^0)y + \text{h.o.t.}
\]

\[
\frac{dy}{dt} = Ay \quad y = 0 \quad \text{the equilibrium point}
\]

\[
A = \frac{df}{dx}(x^0)
\]

**Theorem 1**
If all the eigenvalues of the matrix \( A \) have negative real parts, then the origin is asymptotically stable. If at least one eigenvalue has a positive real part, the origin is unstable.

**Example 1**

\[
\frac{dx}{dt} = f(x)
\]

\[
C \frac{dv_c}{dt} = I_s - f(v_c)
\]

\[
\frac{dv_c}{dt} = -\frac{1}{C} \left( f(v_c) - I_s \right)
\]

\[
\frac{dv_c}{dt} = -\frac{1}{C} i_t
\]

1. \( I_s = 0 \), \( v^0 = 0 \), \( \frac{dv_c}{dt} = -f(v_c) \)
\[ A = -\frac{1}{C} \left. \frac{df}{dv_c} \right|_{v_c = 0} < 0 \]

2. 

\[ I_s = I_1 \]

\[ A_b = -\frac{1}{C} \left. \frac{df}{dv_c} \right|_{v_c = b} < 0 \]

\[ A_a = -\frac{1}{C} \left. \frac{df}{dv_c} \right|_{v_c = a} = 0 \]

\[ v_c = a + \Delta \]

\[ \Delta > 0 : \quad i_1 > 0 \]

\[ \frac{dv_c}{dt} < 0 \]

\[ \Delta < 0 : \quad i_1 > 0 \]

\[ \frac{dv_c}{dt} < 0 \]

\[ v_c = a \quad \text{unstable} \]

3. 

\[ I_s \]

\[ I_1 < I_s < I_2 \]

\[ A_a < 0 \]

\[ A_b > 0 \]

\[ A_e < 0 \]

The direct method of Lyapunov: local stability

Definition 1

Consider a function \( V(x) = V(x_1, ..., x_n) \), defined in some neighborhood \( R \) of the origin. If

(i) \( V(x) \) is continuously differentiable in \( R \),

(ii) \( V(0) = 0 \),
(iii) \( V(x) > 0, \ x \neq 0, x \in R, \)
then \( V(x) \) is said to be positive definite in \( R \)

**Definition 2**
A function \( V(t, x) \), defined in some neighborhood \( R \) of the origin is positive define in \( R \) if

(i) \( V(t, x) \) has continuous derivatives in \( R \) for \( t \geq 0 \),

(ii) \( V(t, 0) = 0, t \geq 0 \)

(iii) there exists a scalar function \( W(x) \), positive definite, such that

\[
W(x) \leq V(t, x), \ x \in R, \ t \geq 0
\]

**Example**
\( (2x_1^2 + 3x_2^2)e^{-t} \) is positive for all \( x \neq 0 \) and for each fixed \( t \), but does not fit the definition of a positive definite function

**Definition 3**
Consider a neighborhood \( R \) of the origin and a function \( V(t, x) \), positive definite in \( R \). \( V(t, x) \) is said to be a Lyapunov function for an equation \( \frac{dx}{dt} = f(t, x) \) if

\[
\frac{dV}{dt} \leq 0 \quad \text{for all } x \in R \text{ and } t \geq 0
\]

where

\[
\frac{dV}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \frac{dx}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x)
\]

**Theorem**
If there exists a neighborhood \( R \) of the origin over which a Lyapunov function can be defined, then the origin is stable

**Example**

![Fig. 6](image-url)
\[
\frac{dv_c}{dt} = -3v_c + i_L
\]
\[
\frac{di_L}{dt} = -\frac{v_c}{2} - \frac{1}{2}i_L
\]

\[-3v_c + i_L = 0\]

\[-\frac{v_c}{2} - \frac{1}{2}i_L = 0\]

\[v_c^{(0)} = 0, \quad i_L^{(0)} = 0\]

\[x = \begin{bmatrix} v_c \\ i_L \end{bmatrix}, \quad V(x) = \frac{1}{2}Cv_c^2 + \frac{1}{2}Li_L^2 = \frac{1}{2}v_c^2 + \frac{1}{2}i_L^2\]

\[
\frac{dv}{dt} = \frac{dV}{dx} \frac{dx}{dt} = [v_c, i_L] \begin{bmatrix} \frac{dv_c}{dt} \\ \frac{di_L}{dt} \end{bmatrix} = [v_c, i_L] \begin{bmatrix} -3v_c + i_L \\ -\frac{v_c}{2} - \frac{1}{2}i_L \end{bmatrix} = -\left(3v_c^2 + \frac{1}{2}i_L^2\right)
\]

\[\forall \ x \neq 0: \frac{dV}{dt} < 0 \quad v_c^{(0)} = 0, \quad i_L^{(0)} = 0 \quad \text{is stable}\]