
Question 1

Describe the circuit shown in Fig. 1 using the node method. Solve the node equation applying the Newton-Raphson method.

Data: \( E_1 = 1 \text{V} \), \( E_2 = 2 \text{V} \), \( R_1 = 1 \Omega \), \( R_2 = 2 \Omega \), \( i = g(v) = 0.5v + 0.1v^3 + 0.05v^5 + 0.01v^7 \), where \( v \) is in volts and \( i \) is amperes.

![Circuit Diagram](image)

**Fig. 1**

**Solution**

We choose the bottom node as a datum one and introduce the node voltage \( e = v \). Then, we write the node equation

\[
\frac{v - E_1}{R_1} + g(v) + \frac{v - E_2}{R_2} = 0 .
\]

(1)

Substituting the parameters yields

\[
2v + 0.1v^3 + 0.05v^5 + 0.01v^7 - 2 = 0 .
\]

The Newton-Raphson method applied to this equation gives

\[
v^{(j+1)} = v^{(j)} - \frac{2v^{(j)} + 0.1(v^{(j)})^3 + 0.05(v^{(j)})^5 + 0.01(v^{(j)})^7 - 2}{2 + 0.3(v^{(j)})^2 + 0.25(v^{(j)})^4 + 0.07(v^{(j)})^6} .
\]

(2)

Let the initial guess be \( v^{(0)} = 0 \). Then we obtain: \( v_1 = 1 \), \( v_2 = 0.939 \), \( v_3 = 0.938 \), \( v_4 = 0.938 \). Thus, the approximate solution is \( v^* = 0.938 \).
Question 2

In the circuit shown in Fig. 2 determine the input voltage \( v_i \) so that \( v_0 = 1 \text{V} \).

![Fig. 2](image)

Data: \( R_1 = 1 \text{k}\Omega, \ v_2 = -1 + i_2 + 0.5i_2^3 + 0.1i_2^5 + e^{0.5v_i} \), where \( v_2 \) is in volts and \( i_2 \) in miliamps, \( E_{\text{sat}} = 14 \text{V} \).

Solution

Since \( 0 < v_0 < E_{\text{sat}} \) the operational amplifier operates in the linear region. Hence, \( v_d = 0 \) and the following equations can be written:

\[
v_0 = -v_2 = -f\left(\frac{v_i}{R_1}\right) = -f(v_i),
\]
\[
v_0 = 1 - v_i - 0.5v_i^3 - 0.1v_i^5 - e^{0.5v_i}.
\]

Substituting \( v_0 = 1 \), the above equation becomes

\[v_i + 0.5v_i^3 + 0.1v_i^5 + e^{0.5v_i} = 0.\] (3)

The Newton-Raphson iteration formula associated to this equation is

\[v_i^{(j+1)} = v_i^{(j)} - \frac{v_i^{(j)} + 0.5(v_i^{(j)})^3 + 0.1(1(v_i^{(j)})^5 + e^{0.5v_i^{(j)})}}{1 + 1.5(v_i^{(j)})^2 + 0.5(v_i^{(j)})^4 + 0.5e^{0.5v_i^{(j)}}}.\] (4)

Let the initial guess be \( v_i^{(0)} = 0 \), then we obtain: \( v^{(1)} = -0.667 \), \( v^{(2)} = -0.614 \), \( v^{(3)} = -0.613 \), \( v^{(4)} = -0.613 \). Thus, the approximate solution is \( v^* = -0.613 \text{V} \).
Question 3

Describe the circuit shown in Fig. 3 using the node approach. Solve the obtained set of equations applying the Newton-Raphson method.

Data: \( R_1 = 2 \, \Omega \), \( R_2 = 4 \, \Omega \), \( i_1 = 1 \, A \), \( i_2 = 3 \, A \), \( i_3 = g_3(v_3) = 2v_3 \), \( i_4 = g_4(v_4) = e^{0.5v_4} \).

![Fig. 3](image)

Solution

We choose the bottom node as a datum one and introduce the node voltages \( e_1 \) and \( e_2 \). Then, we write the node equations

\[
\frac{e_1}{R_1} + 2(e_1 - e_2)^3 - i_{S_1} = 0, \\
-2(e_1 - e_2)^3 + e^{0.5e_2} + \frac{e_2}{R_2} - i_{S_2} = 0.
\]

We substitute the data and obtain

\[
f_1(e_1, e_2) = 0.5e_1 + 2(e_1 - e_2)^3 - 1 = 0, \\
f_2(e_1, e_2) = -2(e_1 - e_2)^3 + e^{0.5e_2} + 0.25e_2 - 3 = 0.
\]

We form the Jacoby matrix

\[
J(e_1, e_2) = \begin{bmatrix}
0.5 + 6(e_1 - e_2)^2 & -6(e_1 - e_2)^2 \\
-6(e_1 - e_2)^2 & 6(e_1 - e_2)^2 + 0.5e^{0.5e_2} + 0.25
\end{bmatrix}
\]

and write the Newton-Raphson equation in the form

\[
J(e_1^{(j)}, e_2^{(j)}) \begin{bmatrix}
y_1^{(j+1)} \\
y_2^{(j+1)}
\end{bmatrix} = -\begin{bmatrix}
f_1(e_1^{(j)}, e_2^{(j)}) \\
f_2(e_1^{(j)}, e_2^{(j)})
\end{bmatrix},
\]

where

\[
\begin{bmatrix}
y_1^{(j+1)} \\
y_2^{(j+1)}
\end{bmatrix} = \begin{bmatrix}
e_1^{(j+1)} - e_1^{(j)} \\
e_2^{(j+1)} - e_2^{(j)}
\end{bmatrix}.
\]
Let the initial guess \( e_1^{(0)}, e_2^{(0)} \) be \( e_1^{(0)} = 0, e_2^{(0)} = 0 \), then

\[
J(e_1^{(0)}, e_2^{(0)}) = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.75 \end{bmatrix}, \quad \begin{bmatrix} f_1(e_1^{(0)}, e_2^{(0)}) \\ f_2(e_1^{(0)}, e_2^{(0)}) \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}
\]

and the Newton-Raphson equation becomes

\[
\begin{bmatrix} 0.5 & 0 \\ 0 & 0.75 \end{bmatrix} \begin{bmatrix} y_1^{(1)} \\ y_2^{(1)} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.
\]

We rewrite this matrix equation as a set of individual equations

\[
0.5y_1^{(1)} = 1,
\]

\[
0.75y_2^{(1)} = 2,
\]

having the solution \( y_1^{(1)} = 2, y_2^{(1)} = 2.667 \). Hence, we find

\[
e_1^{(1)} = y_1^{(1)} + e_1^{(0)} = 2,
\]

\[
e_2^{(1)} = y_2^{(1)} + e_2^{(0)} = 2.667.
\]

Now we set \( j = 1 \) and perform the second iteration. For this purpose we substitute in (8) \( e_1^{(1)} = 2, e_2^{(1)} = 2.667 \)

\[
\begin{bmatrix} 0.5 + 6 \cdot 0.667^2 & -6 \cdot 0.667^2 \\ -6 \cdot 0.667^2 & 6 \cdot 0.667^2 + 0.5e^{0.5 \cdot 2.667} + 0.25 \end{bmatrix} \begin{bmatrix} y_1^{(2)} \\ y_2^{(2)} \end{bmatrix} = \begin{bmatrix} 0.5 \cdot 2 + 2(-0.667)^3 - 1 \\ -2(-0.667)^3 + e^{0.5 \cdot 2.667} + 0.25 \cdot 2.667 - 3 \end{bmatrix}.
\]

After simple manipulations we obtain the matrix equation

\[
\begin{bmatrix} 3.169 & -2.669 \\ -2.669 & 4.816 \end{bmatrix} \begin{bmatrix} y_1^{(2)} \\ y_2^{(2)} \end{bmatrix} = \begin{bmatrix} 0.593 \\ -2.054 \end{bmatrix},
\]

which can be rewritten as a set of two individual equations

\[
3.169y_1^{(2)} - 2.669y_2^{(2)} = 0.593,
\]

\[
-2.669y_1^{(2)} + 4.816y_2^{(2)} = -2.054.
\]

The solution of these equations is \( y_1^{(2)} = -0.322, y_2^{(2)} = -0.604 \). Hence, we find

\[
e_1^{(2)} = e_1^{(1)} + y_1^{(2)} = 1.678,
\]

\[
e_2^{(2)} = e_2^{(1)} + y_2^{(2)} = 2.063.
\]
Question 4

For the set of equation

\[ f_1(x_1, x_2) = x_1 + 2x_1^3 - x_2 + 1 = 0, \]
\[ f_2(x_1, x_2) = -3x_1 + x_2^5 + 2x_2 - 6 = 0, \]  
(11)

carry out three iterations of the Newton-Raphson method. Choose the initial guess \( x_1^{(0)} = 0, \) \( x_2^{(0)} = 1. \)

Solution

We write the Newton-Raphson equation in the form

\[
\begin{bmatrix}
1 + 6(x_1^{(j)})^2 & -1 \\
-3 & 5(x_2^{(j)})^4 + 2
\end{bmatrix}
\begin{bmatrix}
y_1^{(j+1)} \\
y_2^{(j+1)}
\end{bmatrix} =
\begin{bmatrix}
x_1^{(j)} + 2(x_1^{(j)})^3 - x_2^{(j)} + 1 \\
-3x_1^{(j)} + (x_2^{(j)})^5 + 2x_2^{(j)} - 6
\end{bmatrix}.
\]  
(13)

For \( j = 0 \) the equation (13) becomes

\[
\begin{bmatrix}
1 & -1 \\
-3 & 7
\end{bmatrix}
\begin{bmatrix}
y_1^{(0)} \\
y_2^{(0)}
\end{bmatrix} =
\begin{bmatrix}
0 \\
3
\end{bmatrix},
\]
or

\[
y_1^{(0)} - y_2^{(0)} = 0,
\]

\[
-3y_1^{(0)} + 7y_2^{(0)} = 3.
\]  
(14)

The solution of (14) is: \( y_1^{(0)} = 0.75, \) \( y_2^{(0)} = 0.75, \) hence, we have

\[
x_1^{(0)} = x_1^{(0)} + y_1^{(0)} = 0.75, \\
x_2^{(0)} = x_2^{(0)} + y_2^{(0)} = 1.75.
\]  
(15)

To perform the second iteration we assume \( j = 1 \) and substitute into (13) \( x_1^{(1)} \) and \( x_2^{(1)} \) given by (15)

\[
\begin{bmatrix}
4.375 & -1 \\
-3 & 48.894
\end{bmatrix}
\begin{bmatrix}
y_1^{(2)} \\
y_2^{(2)}
\end{bmatrix} =
\begin{bmatrix}
0.843 \\
11.663
\end{bmatrix}.
\]  
(16)

The matrix equation (16) can be rewritten in the form of two equations
\[ 4.375y_1^{(2)} - y_2^{(2)} = -0.843, \]
\[ -3y_1^{(2)} + 48.894y_2^{(2)} = -11.663, \]

having the solution \( y_1^{(2)} = -0.250, \ y_2^{(2)} = -0.254. \)

Hence, we find
\[ x_1^{(2)} = x_1^{(1)} + y_1^{(2)} = 0.5, \]
\[ x_2^{(2)} = x_2^{(1)} + y_2^{(2)} = 1.496. \]

Similarly, for \( j = 3 \) we obtain
\[
\begin{bmatrix}
2.5 & -1 \\
-3 & 27 \\
\end{bmatrix}
\begin{bmatrix}
y_1^{(3)} \\
y_2^{(3)} \\
\end{bmatrix} =
\begin{bmatrix}
-0.254 \\
-2.985 \\
\end{bmatrix},
\]
or
\[
2.5y_1^{(3)} - y_2^{(3)} = -0.254, \]
\[-3y_1^{(3)} + 27y_2^{(3)} = -2.985. \quad (17) \]

The solution of the system of equations (17) is \( y_1^{(3)} = -0.153, \ y_2^{(2)} = -0.127. \) Hence, it follows
\[ x_1^{(3)} = x_1^{(2)} + y_1^{(3)} = 0.347, \]
\[ x_2^{(3)} = x_2^{(2)} + y_2^{(3)} = 1.369. \]
Question 5

In the circuit shown in Fig. 4 replace the bipolar transistor by the simplified Ebers-Moll model consisting of the emitter diode and the current controlled current source (see Fig. 5), where $i_d = K(e^{\lambda v_d} - 1)$.

For the circuit shown in Fig. 5 create the Newton-Raphson discrete equivalent circuit and describe it using the node method.

Solution

First we form the discrete Newton-Raphson model of the diode described by the equation

$$i_d = K(e^{\lambda v_d} - 1).$$  \hspace{1cm} (18)

Using the main idea of Newton-Raphson method we write

$$i_d^{(j+1)} = i_d^{(j)} + \frac{di_d}{dv_d}(v_d^{(j)}) (v_d^{(j+1)} - v_d^{(j)}) = I^{(j)} + G^{(j)} v_d^{(j+1)},$$  \hspace{1cm} (19)

where

$$G^{(j)} = \frac{di_d}{dv_d}(v_d^{(j)}) = \lambda K e^{\lambda v_d^{(j)}},$$  \hspace{1cm} (20)

$$I^{(j)} = i_d^{(j)} - \lambda K e^{\lambda v_d^{(j)}} v_d^{(j)}.$$  \hspace{1cm}

The equation (19) enable us to form the discrete Newton-Raphson model of the diode, shown in Fig. 6.
The diode model enables us to create the Newton-Raphson discrete equivalent circuit of the circuit shown in Fig. 5, modeling this circuit at \((j+1)\)-st iteration (see Fig. 7).

The node equations describing the circuit shown in Fig. 7 are as follows

\[
\frac{1}{R_2}e_1^{(j+1)} + \frac{1}{R_1}(e_1^{(j+1)} - e_3^{(j+1)}) + (1 - \alpha)(G^{(j)}(e_1^{(j+1)} - e_2^{(j+1)}) + I^{(j)}) = 0, \\
\frac{1}{R_E}e_2^{(j+1)} + G^{(j)}(e_2^{(j+1)} - e_1^{(j+1)}) - I^{(j)} = 0, \\
\frac{1}{R_1}(e_3^{(j+1)} - e_1^{(j+1)}) + \alpha G^{(j)}(e_1^{(j+1)} - e_2^{(j+1)}) + \alpha I^{(j)} + \frac{1}{R_L}e_3^{(j+1)} - \frac{E}{R_L} = 0.
\]