The Newton-Raphson method

The analysis of nonlinear resistive circuits requires the solution of systems of nonlinear algebraic equations. The most powerful numerical algorithm enabling us to solve the system of equations is the Newton-Raphson one. To explain it we consider at first the simplest case of a single equation in a single variable

$$f(x) = 0. \tag{1}$$

The Newton-Raphson method is an iterative method and the iteration formula is

$$x^{(j+1)} = x^{(j)} - \frac{f\left(x^{(j)}\right)}{f'\left(x^{(j)}\right)}, \tag{2}$$

where

$$f'\left(x^{(j)}\right) = \frac{df}{dx}\left(x^{(j)}\right). \tag{3}$$

Starting with an initial guess $x^{(0)}$ we compute $x^{(1)}$ from the equation (2). We use this equation successively until $x^{(j)}$ converges to the solution x^*. The Newton-Raphson method has the following geometric interpretation.
We write the formula (2) in the from

$$f\left(x^{(j)}\right) + f'\left(x^{(j)}\right)\left(x^{(j+1)} - x^{(j)}\right) = 0 \tag{4}$$

and consider the function

$$y = f\left(x^{(j)}\right) + f'\left(x^{(j)}\right)\left(x - x^{(j)}\right). \tag{5}$$

The equation (5) describes a straight line passing through the point $\left(x^{(j)}\right), f\left(x^{(j)}\right)$ with the slope $f'\left(x^{(j)}\right).$ Thus, the straight line is the tangent to the curve $y = f(x)$. For $x = x^{(j+1)}$ it holds $y = 0$. Hence, $x^{(j+1)}$ is a point of intersection of the tangent with the 0x axis. The geometric interpretation is shown in Fig. 1
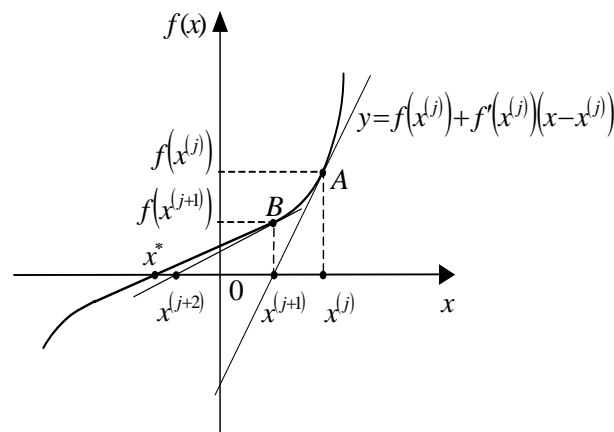

Fig. 1

The Newton-Raphson computation process is continued until $x^{(j+1)} - x^{(j)} \cong 0$. Since

$$x^{(j+1)} - x^{(j)} = \frac{f\left(x^{(j)}\right)}{f'\left(x^{(j)}\right)}$$

then the equation

$$x^{(j+1)} - x^{(j)} = 0$$

implies

$$f\left(x^{(j)}\right) = 0.$$

Thus, we assume

$$x^{(j)} \cong x^*.$$

Example 1

Given the equation

$$f(x) = x^3 + x - 4 = 0.$$

To writhe the Newton-Raphson iteration formula adapted to this equation we find the derivative

$$f'(x) = 3x^2 + 1.$$

Hence, we have

$$x^{(j+1)} = x^{(j)} - \frac{\left(x^{(j)}\right)^3 + x^{(j)} - 4}{3\left(x^{(j)}\right)^2 + 1}.$$

We choose arbitrarily an initial guess, say $x^{(0)} = 0$.
The Newton-Raphson method generates the sequence

$$x^{(1)} = 0 - \frac{-4}{1} = 4,$$

$$x^{(2)} = 4 - \frac{64 + 4 - 4}{48 + 1} = 2.694,$$

$$x^{(3)} = 2.694 - \frac{2.694^3 + 2.694 - 4}{3 \cdot 2.694^2 + 1} = 1.893,$$

$$x^{(4)} = 1.495,$$
$$x^{(5)} = 1.386,$$
$$x^{(6)} = 1.379,$$
$$x^{(7)} = 1.379.$$

Since $x^{(7)} - x^{(6)} = 0$, we assume $x^* = 1.379$.

Convergence of the Newton-Raphson method

Assume that $x^{(j)}$ is near x^* and expand $f(x)$ about $x^{(j)}$ in the Taylor series. Then we have

$$f\left(x^*\right)= f\left(x^{(j)}\right)+ f'\left(x^{(j)}\right)\left(x^* - x^{(j)}\right)+\gamma\left(x^*, x^{(j)}\right) \tag{6}$$

where

$$\left|\gamma\left(x^*, x^{(j)}\right)\right|\le \alpha\left|x^* - x^{(j)}\right|^2 \tag{7}$$

and $f\left(x^*\right)=0$. We subtract the equation (4), repeated below

$$f\left(x^{(j)}\right)+ f'\left(x^{(j)}\right)\left(x^{(j+1)} - x^{(j)}\right)= 0$$

from the equation (6). As a result we obtain

$$f'\left(x^{(j)}\right)\left(x^{(j+1)} - x^*\right)= \gamma\left(x^*, x^{(j)}\right).$$

Let us assume that $f'\left(x^{(*)}\right)\ne 0$. Then for $x^{(j)}$ sufficiently close to x^*, there exists $\beta > 0$ such that

$$\frac{1}{\left|f'\left(x^{(j)}\right)\right|}\le \beta. \tag{9}$$

Using (8), (9) and (7) we find

$$\left|x^{(j+1)} - x^*\right|\le \beta\left|\gamma\left(x^*, x^{(j)}\right)\right|\le \beta\alpha\left|x^{(j)} - x^*\right|^2. \tag{10}$$

The relation (10) states that the rate of convergence of the Newton-Raphson method is quadratic.

Convergence problem

According to the obove discussion the Newton-Raphson method works when the initial guess is sufficiently near the solution and the function is well-behaved. If the initial guess is far off from x^* or if $f(x)$ is not smooth the method may not converge (see Figs. 2 and 3).
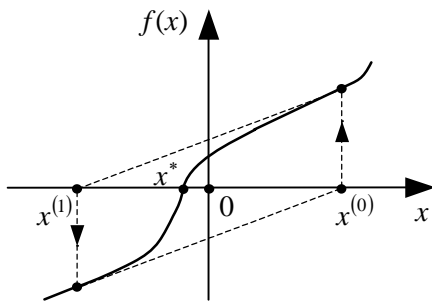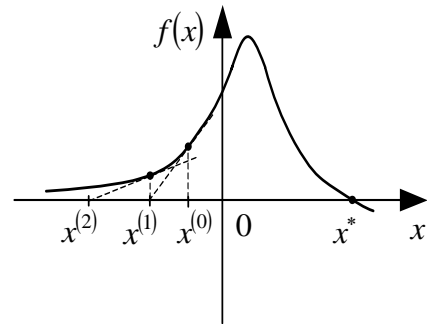

Fig. 2

Fig. 3

Example 2

Given the circuit shown in Fig. 4, where the diode is described by the equation
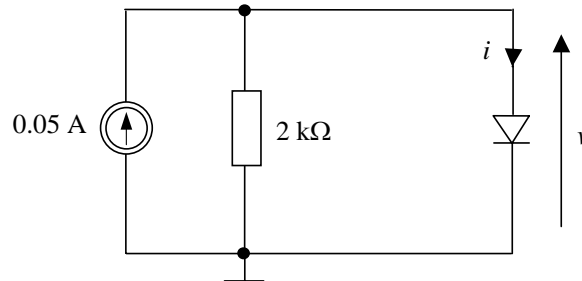
$$i = 10^{-15}\left(e^{38v} - 1\right).$$

Fig. 4

We write the node equation

$$0.0005\,v + 10^{-15}\left(e^{38v} - 1\right) - 0.05 = 0.$$

The Newton-Raphson iteration formula adapted to this equation is

$$v^{(j+1)} = v^{(j)} - \frac{0.0005\,v^{(j)} + 10^{-15}\left(e^{38v^{(j)}} - 1\right) - 0.05}{0.0005 + 38 \cdot 10^{-15}\,e^{38v^{(j)}}}.$$

Let $v^{(0)} = 0$, then $v^{(1)} = \dfrac{0.05}{0.0005} = 100$ and

$$v^{(2)} = 100 - \frac{0.05 + 10^{-15}\left(e^{3800} - 1\right) - 0.05}{0.0005 + 38 \cdot 10^{-15}\,e^{3800}} \cong 100 - \frac{10^{-15}\,e^{3800}}{38 \cdot 10^{-15}\,e^{3800}} = 99.9737.$$

Since the slop of the tangent at $v^{(1)}$ is very steep $v^{(2)}$ is very close to $v^{(1)}$. Consequently huge number of iterations is necessary to find the solution $v^* = 0.8299$.

To improve the method modify the diode characteristic as follows

$$i = \begin{cases} 10^{-15}\left(e^{38V} - 1\right) & \text{if} \quad v \le 0.9 \\ 10^{-15}\left(e^{38 \cdot 0.9} - 1\right) + 38 \cdot 10^{-15} e^{38 \cdot 0.9}\left(v - 0.9\right) & \text{if} \quad > 0.9. \end{cases}$$

Thus, the diode described by the original function for $v \le 0.9$ and by the linear function describing the tangent passing through the point $\left(0.9,\ 10^{-15}\left(e^{38 \cdot 0.9} - 1\right)\right)$ for $v > 0.9$. This modification considerably speed up the method, which leads to the solution in several iterations.

General version of the Newton-Raphson method

This section describes the Newton-Raphson method for higher dimensions. Let us consider an equation $f(x) = 0$, where

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix}, \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The solution to the equation will be labeled

$$\mathbf{x}^* = \begin{bmatrix} x_1^* & x_2^* & \cdots & x_n^* \end{bmatrix}^{\mathrm{T}}.$$

In this case the Newton-Raphson formula is

$$x^{(j+1)} = x^{(j)} - J^{-1}(x^{(j)}) f(x^{(j)}) \tag{11}$$

where $J(x(j))$ is the Jacoby matrix

$$J(x^{(j)}) = \begin{bmatrix} J_{11}(x^{(j)}) & J_{12}(x^{(j)}) & \cdots & J_{1n}(x^{(j)}) \\ J_{21}(x^{(j)}) & J_{22}(x^{(j)}) & \cdots & J_{2n}(x^{(j)}) \\ \cdots & \cdots & \cdots & \cdots \\ J_{n1}(x^{(j)}) & J_{n2}(x^{(j)}) & \cdots & J_{nn}(x^{(j)}) \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1}(x^{(j)}) & \dfrac{\partial f_1}{\partial x_2}(x^{(j)}) & \cdots & \dfrac{\partial f_1}{\partial x_n}(x^{(j)}) \\ \dfrac{\partial f_2}{\partial x_1}(x^{(j)}) & \dfrac{\partial f_2}{\partial x_2}(x^{(j)}) & \cdots & \dfrac{\partial f_2}{\partial x_n}(x^{(j)}) \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial f_n}{\partial x_1}(x^{(j)}) & \dfrac{\partial f_n}{\partial x_2}(x^{(j)}) & \cdots & \dfrac{\partial f_n}{\partial x_n}(x^{(j)}) \end{bmatrix}. \tag{12}$$

Equation (11) can be presented in the equivalent form

$$J(x^{(j)}) y^{(j+1)} = -f(x^{(j)}), \tag{13}$$

where

$$\mathbf{y}^{(j+1)} = \mathbf{x}^{(j+1)} - \mathbf{x}^{(j)}.$$

Using equation (13) we do not need to determine the inversion of the Jacoby matrix which is a time consuming task. In a special case, where $n = 2$, the Jacoby matrix becomes

$$J(x^{(j)}) = \begin{bmatrix} J_{11}(x^{(j)}) & J_{12}(x^{(j)}) \\ J_{21}(x^{(j)}) & J_{22}(x^{(j)}) \end{bmatrix},$$

where

$$J_{11}(x_1, x_2) = \frac{\partial f_1}{\partial x_1}, \qquad\qquad J_{12}(x_1, x_2) = \frac{\partial f_1}{\partial x_2},$$

$$J_{21}(x_1, x_2) = \frac{\partial f_2}{\partial x_1}, \qquad\qquad J_{22}(x_1, x_2) = \frac{\partial f_2}{\partial x_2},$$

$$x^{(j)} = \begin{bmatrix} x_1^{(j)} \\ x_2^{(j)} \end{bmatrix}, \qquad x^{(j+1)} = \begin{bmatrix} x_1^{(j+1)} \\ x_2^{(j+1)} \end{bmatrix}, \qquad f(x^{(j)}) = \begin{bmatrix} f_1(x^{(j)}) \\ f_2(x^{(j)}) \end{bmatrix}.$$

Example 3

Let us consider the set of two equations

$$0.1x_1^3 + 2x_1 - 4x_2 - 5 = 0,$$
$$-x_1 + 0.02x_2^5 + 0.5x_2^3 + x_2 - 12 = 0.$$

To find the solution using the Newton-Raphson method we compute the Jacoby matrix

$$\boldsymbol{J}(\boldsymbol{x}) = \begin{bmatrix} 0.3x_1^2 + 2 & -4 \\ -1 & 0.1x_2^4 + 1.5x_2^2 + 1 \end{bmatrix}$$

and white the Newton-Raphson equation (13)

$$\begin{bmatrix} 0.3\left(x_1^{(j)}\right)^2 + 2 & -4 \\ -1 & 0.1\left(x_2^{(j)}\right)^4 + 1.5\left(x_2^{(j)}\right)^2 + 1 \end{bmatrix}\begin{bmatrix} y_1^{(j+1)} \\ y_2^{(j+1)} \end{bmatrix} =$$
$$-\begin{bmatrix} 0.1\left(x_1^{(j)}\right)^3 + 2x_1^{(j)} - 4x_2^{(j)} - 5 \\ -x_1^{(j)} + 0.02\left(x_2^{(j)}\right)^5 + 0.5\left(x_2^{(j)}\right)^3 + x_2^{(j)} - 12 \end{bmatrix},$$

where

$$\begin{bmatrix} y_1^{(j+1)} \\ y_2^{(j+1)} \end{bmatrix} = \begin{bmatrix} x_1^{(j+1)} \\ x_2^{(j+1)} \end{bmatrix} - \begin{bmatrix} x_1^{(j)} \\ x_2^{(j)} \end{bmatrix}.$$

Choosing the initial guess $x^{(0)} = \begin{bmatrix} 5 & 5 \end{bmatrix}^T$ we find

$$\boldsymbol{x}^{(1)} = \begin{bmatrix} 4.7912 \\ 3.8791 \end{bmatrix}, \qquad \boldsymbol{x}^{(2)} = \begin{bmatrix} 4.4510 \\ 3.1395 \end{bmatrix}, \qquad \boldsymbol{x}^{(3)} = \begin{bmatrix} 4.2638 \\ 2.8082 \end{bmatrix},$$

$$\boldsymbol{x}^{(4)} = \begin{bmatrix} 4.2252 \\ 2.7479 \end{bmatrix}, \qquad \boldsymbol{x}^{(5)} = \begin{bmatrix} 4.2240 \\ 2.7461 \end{bmatrix}, \qquad \boldsymbol{x}^{(6)} = \begin{bmatrix} 4.2240 \\ 2.7461 \end{bmatrix}.$$

Since $\boldsymbol{x}^{(6)} - \boldsymbol{x}^{(5)} = 0$, we assume $\boldsymbol{x}^* = \boldsymbol{x}^{(5)}$.

Discrete models

To improve the Newton-Raphson method we introduce an associated linear resistive circuit at each iteration, called the Newton-Raphson discrete equivalent circuit. To explain how this circuit is formed we consider a nonlinear voltage controlled resistor described by the equation

$$i = f(v).$$

According to the idea of the Newton-Raphson method the nonlinear function $f(v)$ is replaced at $(j+1)$-st iteration by the linear function

$$i = i^{(j)} + f'\left(v^{(j)}\right)\left(v - v^{(j)}\right), \tag{14}$$

or

$$i = \left(i^{(j)} - f'\left(v^{(j)}\right)v^{(j)}\right) + f'\left(v^{(j)}\right)v.$$

Thus, $(j+1)$-st iterate satisfies the equation

$$i^{(j+1)} = \left(i^{(j)} - f'\left(v^{(j)}\right)v^{(j)}\right) + f'\left(v^{(j)}\right)v^{(j+1)}. \tag{15}$$

Let us denote

$$\hat{i}^{(j)} = i^{(j)} - f'\left(v^{(j)}\right)v^{(j)} \tag{16}$$

and

$$G^{(j)} = f'\left(v^{(j)}\right). \tag{17}$$

Then the equation (15) becomes

$$i^{(j+1)} = \hat{i}^{(j)} + G^{(j)}v^{(j+1)}. \tag{18}$$

The circuit described by the equation (18) called the discrete Newton-Raphson model of the resistor is shown in Fig.5.
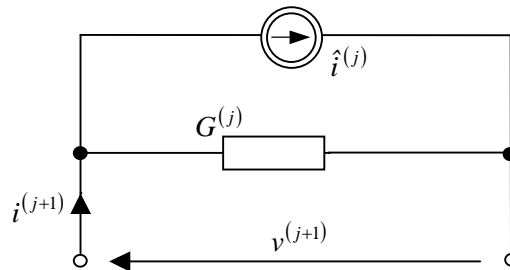

Fig. 5

Replacing all nonlinear elements by the discrete models we obtain a linear circuit. Its solution corresponds to the $(j+1)$ iteration.

Example

Given the circuit shown in Fig. 6, where the diode is specified by the equation $i = K\left(e^{\lambda v} - 1\right)$.
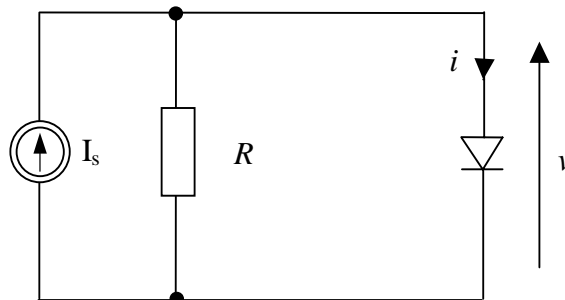

Fig. 6

We create the discrete Newton-Raphson model of the diode. The equation which describes this model is

$$i^{(j+1)} = i^{(j)} + \lambda \, K \, e^{\lambda v^{(j)}} \left(v^{(j+1)} - v^{(j)} \right)$$

or

$$i^{(j+1)} = \left(i^{(j)} - \lambda \, K \, e^{\lambda v^{(j)}} v^{(j)} \right) + \lambda \, K \, e^{\lambda v^{(j)}} v^{(j+1)}.$$

Substituting

$$G^{(j)} = \lambda \, K \, e^{\lambda u^{(j)}} \qquad \text{and} \qquad I^{(j)} = i^{(j)} - G^{(j)} v^{(j)},$$

we obtain $i^{(j+1)} = I^{(j)} + G^{(j)} v^{(j+1)}$.
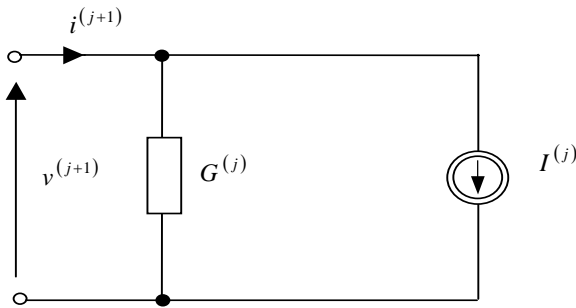Hence the discrete Newton-Raphson model of the diode shown in Fig. 7.

Fig. 7

We introduce this model into the circuit and from the circuit corresponding to $(j+1)$-st iteration (see Fig. 8).
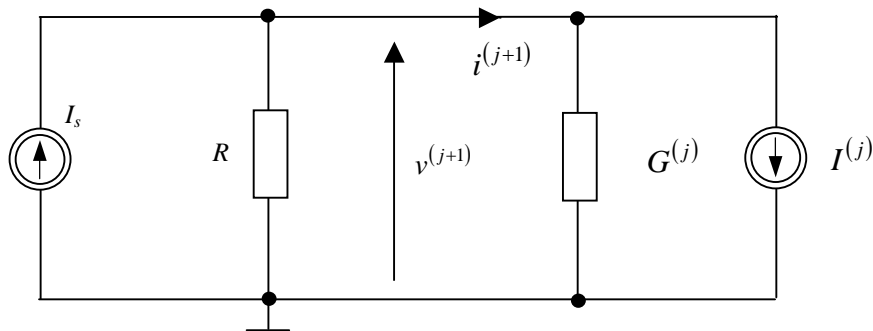

Fig. 8

To solve the circuit shown in Fig.8 we apply the node approach. The voltage $v^{(j+1)}$ is the node voltage. We write the node equation

$$\frac{v^{(j+1)}}{R} + G^{(j)} v^{(j+1)} + I^{(j)} - I_s = 0 \tag{19}$$

and solve it finding the $(j+1)$ iterate

$$v^{(j+1)} = \frac{I_s - K\left(e^{\lambda v^{(j)}} - 1\right) + \lambda K e^{\lambda v^{(j)}} v^{(j)}}{\dfrac{1}{R} + \lambda K e^{\lambda v^{(j)}}}. \tag{20}$$

Alternatively, we formulate the node equation of the circuit shown in Fig. 6

$$K\left(e^{\lambda v} - 1\right) + \frac{1}{R} v - I_s = 0.$$

Using the Newton-Raphson iteration formula we find

$$v^{(j+1)} = v^{(j)} - \frac{K\left(e^{\lambda v^{(j)}} - 1\right) + \dfrac{1}{R} v^{(j)} - I_s}{\lambda K e^{\lambda v^{(j)}} + \dfrac{1}{R}} =$$

$$= \frac{\lambda K e^{\lambda v^{(j)}} v^{(j)} + \dfrac{1}{R} v^{(j)} - K\left(e^{\lambda v^{(j)}} - 1\right) - \dfrac{1}{R} v^{(j)} + I_s}{\lambda K e^{\lambda v^{(j)}} + \dfrac{1}{R}} =$$

$$= \frac{I_s - K\left(e^{\lambda v^{(j)}} - 1\right) + \lambda K e^{\lambda v^{(j)}} v^{(j)}}{\dfrac{1}{R} + \lambda K e^{\lambda v^{(j)}}}.$$

The result is the some as given by the equation (20).