13. The *Z*-transform

13.1. Introduction

The *Z*-transform is a mathematical operation that transforms a sequence of numbers representing a discrete-time signal into a function of a complex variable. The *Z*-transform can be considered as an equivalent of the Laplace transform applicable to discrete systems as follows.

Let $x(t)$ be a continuous signal and $\tilde{x}(t)$ an associated signal obtained by sampling $x(t)$ with an interval T and multiplying the samples by the unit impulses

$$\tilde{x}(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta\left(t - nT\right). \tag{13.1}$$

We compute the Laplace transform of $\tilde{x}(t)$

$$\tilde{X}(s) = \sum_{n=-\infty}^{\infty} x(nT)\mathcal{L}\left(\delta\left(t - nT\right)\right). \tag{13.2}$$

Using the shifting theorem (see Section 2.2.5), we obtain

$$\mathcal{L}\left(\delta\left(t - nT\right)\right) = \begin{cases} 0 & \text{for} \quad n < 0 \\ \mathrm{e}^{-snT} & \text{for} \quad n \geq 0 \end{cases}. \tag{13.3}$$

Hence, we have

$$\tilde{X}(s) = \sum_{n=0}^{\infty} x\left(nT\right)\mathrm{e}^{-snT} \;. \tag{13.4}$$

To derive the *Z*-transform of discrete signal $x(nT)$ we substitute in (13.4)

$$z = \mathrm{e}^{sT} \tag{13.5}$$

and denote the left hand side of the equation by $X(z)$. Furthermore, we write $x(n)$ instead of $x(nT)$, obtaining

$$X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n} \tag{13.6}$$

where $X(z)$ is called the *Z*-transform of the sequence $x(n)$. The operation of the *Z*-transform is labeled *Z*:

$$Z(x(n)) = X(z).$$

The operation of going back to the sequence $x(n)$ from its *Z*-transform $X(z)$ is labeled Z^{-1}

$$x(n) = Z^{-1}(X(z)).$$

Expression (13.6) is an infinite series. In a special case where the signal $x(n)$ consists only of a finite number of points the series reduces to

$$X(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + \ldots + x(N)z^{-N}. \tag{13.7}$$

The *Z*-transform of a signal $x(n)$ exists if the infinite sum (13.6) converges. The region of convergence is defined by the following theorem.
Let us assume that $x(n)$ is absolutely summable over the interval $0 \le n \le M$ for any $M > 0$, that is

$$\sum_{n=0}^{M} |\, x(n)| < \infty$$

and such real parameters *r* and *A* can be chosen that

$$\left| r^{-n} x(n) \right| \le A \qquad \text{for} \qquad n \ge M .$$

Then the sum (13.6) converges absolutely for all z such that $|z| > r$.

The condition $|z| > r$ means that z is located in the complex plane outside the circle with the radius r. Thus, all z such that $|z| > r$ belong to the region of existence of the *Z*-transform.

Example 13.1

Let us consider the unit sample

$$\delta(n) = \begin{cases} 1 & \text{for} \quad n = 0 \\ 0 & \text{otherwise} \end{cases}. \tag{13.8}$$

Hence, only the first term in (13.7) exists and

$$Z(\delta(n)) = 1. \tag{13.9}$$

Example 13.2

Let us consider the discrete signal

$$\{x(n)\} = \{1, a, a^2, \dots\}.$$

Its *Z*-transform contains infinitely many terms

$$X(z) = 1 + az^{-1} + a^2 z^{-2} + \dots = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n. \tag{13.10}$$

The expression (13.10) is a geometric series with the zeroth term equal to one and ratio $q = \frac{a}{z}$. If $|q| < 1$ or equivalently $|z| > |a|$ the series converges to

$$X(z) = \frac{1}{1 - \frac{a}{z}} = \frac{z}{z - a}. \tag{13.11}$$

Thus, the *Z*-transform of the sequence $\{a^n\}$ exists for $|z| > |a|$ and the region in the complex plane where $|z| > |a|$ defines the region of existence of the *Z*-transform as it is illustrated in Fig.13.1.

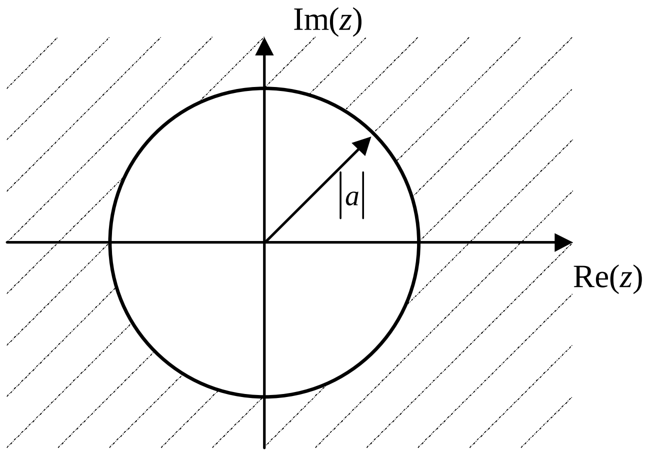

Fig. 13.1. The region of existence of the *Z*-transform for Example 13.1

If $a = 1$ then the signal becomes the unit step discrete signal

$$u(n) = \begin{cases} 1 & \text{for} \quad n \geq 0 \\ 0 & \text{otherwise} \end{cases}. \tag{13.12}$$

Setting $a = 1$ in (13.11) yields

$$Z(u(n)) = \frac{z}{z-1}. \tag{13.13}$$

13.2. Properties

Linearity property

If $x_1(n)$ and $x_2(n)$ are two signals having the *Z*-transforms $X_1(z)$ and $X_2(z)$ respectively, and c_1 and c_2 are arbitrary constants, then

$$Z(c_1 x_1(n) + c_2 x_2(n)) = c_1 X_1(z) + c_2 X_2(z). \tag{13.14}$$

Proof

$$Z\left(c_1x_1(n)+c_2x_2(n)\right)=\sum_{n=0}^{\infty}\left(c_1x_1(n)+c_2x_2(n)\right)z^{-n}=$$
$$=c_1\sum_{n=0}^{\infty}x_1(n)z^{-n}+c_2\sum_{n=0}^{\infty}x_2(n)z^{-n}=c_1X_1(z)+c_2X_2(z).$$

Example 13.3

Let us consider the signal

$$x(n)=(0.3)^n-2(0.5)^n.$$

Using equation (13.11) and the linearity property, we obtain

$$X(z)=\frac{z}{z-0.3}-2\frac{z}{z-0.5}=\frac{-z\,(z-0.1)}{(z-0.3)\,(z-0.5)}.$$

Differentiation rule

The derivative of the Z-transform of a signal $x(n)$ multiplied by $(-z)$ equals the Z-transform of this signal multiplied by n

$$-z\frac{\mathrm{d}X(z)}{\mathrm{d}z}=Z(nx(n)). \tag{13.15}$$

Proof

Since $X(z)=\sum_{n=0}^{\infty}x(n)\,z^{-n}$ then we may write

$$\frac{\mathrm{d}X(z)}{dz}=-\sum_{n=0}^{\infty}x(n)nz^{-n-1}.$$

Multiplying both sides of the above equation by $(-z)$ gives

$$-z\frac{\mathrm{d}X(z)}{\mathrm{d}z}=\sum_{n=0}^{\infty}x(n)nz^{-n}=Z(nx(n)).$$

Example 13.4

Let us consider the discrete unit ramp given by

$$x(n)=\{0,1,2,3,\dots\}.$$

To find the *Z*-transform, we write

$$x(n)=nu(n)$$

where $u(n)$ is the discrete unit step

$$u(n)=\begin{cases}0 & \text{for} \quad n<0\\ 1 & \text{for} \quad n\geq 0\end{cases}$$

and apply the differentiation rule

$$Z\left(x(n)\right)=Z\left(nu(n)\right)=-z\frac{\mathrm{d}}{\mathrm{dz}}Z\left(u(n)\right).$$

Using equation

$$Z(u(n))=\frac{z}{z-1}$$

we obtain

$$X(z)=Z(x(n))=\frac{z}{(z-1)^2}. \qquad (13.16)$$

Since the series nz^{-n} is convergent for $|z|>1$, the region of existence of the *Z*-transform of the discrete unite ramp is defined by inequality $|z|>1$ (see Fig.13.1 with $|a|=1$).

Let us consider the discrete exponential signal given by

$$x(n)=\left\{\mathrm{e}^{-anT}\right\} \qquad (13.17)$$

where *T* is the sampling interval. Since $\mathrm{e}^{-anT}=\left(\mathrm{e}^{-aT}\right)^n$, the *Z*-transform of this signal can be found using (13.11) where *a* should be replaced by e^{-aT}.
Thus,

$$Z\left(\mathrm{e}^{-anT}\right) = \frac{z}{z - \mathrm{e}^{-aT}} \tag{13.18}$$

holds where the region of existence of the *Z*-transform is defined by the inequality

$$\left|\mathrm{e}^{aT} z\right| > 1\,.$$

Using above result we can determine the *Z*-transform of the functions $\cos n\omega T$ and $\sin n\omega T$ as follows

$$\cos n\omega T = \frac{1}{2}\left(\mathrm{e}^{\mathrm{j}n\omega T} + \mathrm{e}^{-\mathrm{j}n\omega T}\right)$$

$$\begin{aligned} Z\left(\cos n\omega T\right) &= \frac{1}{2}\left(Z\left(\mathrm{e}^{\mathrm{j}n\omega T}\right) + Z\left(\mathrm{e}^{-\mathrm{j}n\omega T}\right)\right) = \frac{1}{2}\left(\frac{z}{z-\mathrm{e}^{\mathrm{j}\omega T}} + \frac{z}{z-\mathrm{e}^{-\mathrm{j}\omega T}}\right) = \\ &= \frac{z\left(z - \cos\omega T\right)}{z^2 - 2z\cos\omega T + 1} \end{aligned} \tag{13.19}$$

where the region of existence of the *Z*-transform is $|z| > \left|\mathrm{e}^{\pm\mathrm{j}\omega T}\right| = 1$.
Similarly we determine the *Z*-transform of the function $\sin n\omega T$

$$\sin n\omega T = \frac{1}{2\mathrm{j}}\left(\mathrm{e}^{\mathrm{j}n\omega T} - \mathrm{e}^{-\mathrm{j}n\omega T}\right)$$

$$Z\left(\sin n\omega T\right) = \frac{1}{2\mathrm{j}}\left(\frac{z}{z-\mathrm{e}^{\mathrm{j}\omega T}} - \frac{z}{z-\mathrm{e}^{-\mathrm{j}\omega T}}\right) = \frac{z\sin\omega T}{z^2 - 2z\cos\omega T + 1} \tag{13.20}$$

where $|z| > 1$.

Shifting property

Let us consider a sequence $x(n)$ as depicted in Fig.13.2.

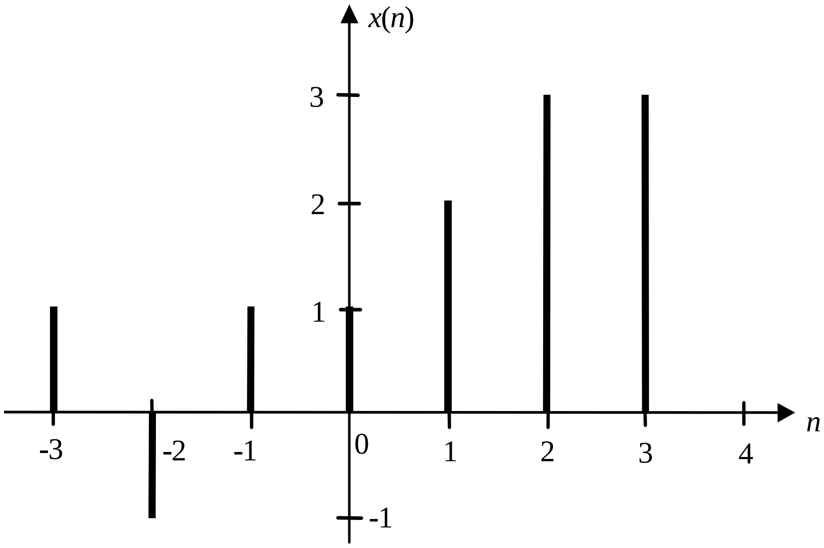

Fig. 13.2. An example sequence

As usual, we label its Z-transform $X(z)$. Next we form the sequence delayed by k sample points $x(n-k)$. Fig.13.3 shows the delayed sequence with $k = 2$.

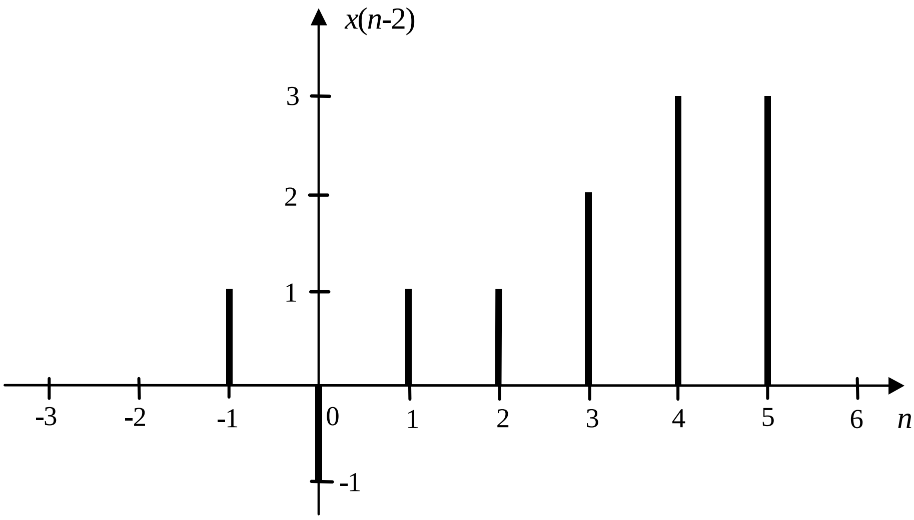

Fig. 13.3. The sequence of Figure 13.2 delayed by two units

The Z-transform of the delayed sequence is

$$Z(x(n-k)) = \sum_{n=0}^{\infty} x(n-k) z^{-n} \qquad k > 0 . \tag{13.21}$$

The expression on the right hand side of equation (13.21) can be rearranged as follows

$$\begin{aligned}\sum_{n=0}^{\infty} x(n-k) z^{-n} &= \sum_{n=0}^{\infty} x(n-k) z^{-k} z^{-(n-k)} = \sum_{m=-k}^{\infty} x(m) z^{-k} z^{-m} = \\ &= z^{-k} \sum_{m=-k}^{-1} x(m) z^{-m} + z^{-k} \sum_{m=0}^{\infty} x(m) z^{-m} = \\ &= x(-k) + x(-k+1) z^{-1} + \ldots + x(-1) z^{-(k-1)} + z^{-k} X(z) .\end{aligned}$$

Thus, we may write

$$Z(x(n-k)) = x(-k) + x(-k+1) z^{-1} + \ldots + x(-1) z^{-(k-1)} + z^{-k} X(z) . \tag{13.22}$$

For the sequence shown in Fig.13.2 we have

$$Z(x(n-2)) = -1 + z^{-1} + z^{-2} X(z) .$$

In a special case where the signal has the property $x(n) = 0$ for $n < 0$ the Z-transform of the delayed sequence is

$$Z(x(n-k)) = z^{-k} X(z) . \tag{13.23}$$

Final-value theorem

If $x(n)$ has the Z-transform $X(z)$ and the singularities of $\frac{z-1}{z} X(z)$, i.e. points at which $\frac{z-1}{z} X(z)$ approaches ∞, are inside the unit circle, then

$$x(\infty) = \lim_{z \to 1} \frac{z-1}{z} X(z) . \tag{13.24}$$

Note that this theorem holds under the assumption that all singularities of $X(z)$ are inside the unit circle, except for possibly one at $z=1$. Furthermore, the final value of $x(n)$ is nonzero if and only if a singularity exists at $z=1$.

Example 13.5

Find the final value of the signal having the following *Z*-transform

$$X(z)=\frac{z(z+2)}{(z-0.5)(z-0.25)}.$$

We consider the expression

$$\frac{z-1}{z}X(z)=\frac{(z-1)(z+2)}{(z-0.5)(z-0.25)}$$

and state that its singularities are inside the unit circle. Thus, we can apply equation (13.24)

$$x(\infty)=\lim_{z\to 1}\frac{(z-1)(z+2)}{(z-0.5)(z-0.25)}=0.$$

As another example we consider signal $y(n)$ having the *Z*-transform

$$Y(z)=\frac{z+1}{(z-1)(z-0.5)}.$$

Since the function

$$\frac{z-1}{z}Y(z)=\frac{z+1}{z(z-0.5)}$$

has singularities only inside the unit circle, we have

$$y(\infty)=\lim_{z\to 1}\frac{z+1}{z(z-0.5)}=4.$$

As a final example, we consider a signal

$$g(n) = -0.4 \cdot 0.5^n + 0.4 \cdot 3^n$$

having the *Z*-transform

$$G(z) = \frac{z}{(z-0.5)(z-3)}\ .$$

The function

$$\frac{z-1}{z} G(z) = \frac{z-1}{(z-0.5)(z-3)}$$

has a singularity outside the unit circle and the final value theorem cannot be applied. Direct substitution gives $g(\infty) \to \infty$ whereas the expression

$$\lim_{z\to 1} \frac{z-1}{(z-0.5)(z-3)}$$

gives errorneous value 0.

13.3. Inversion of the *Z*-transform

Suppose that we have the *Z*-transform of a certain signal $x(n)$ and we wish to find the signal $x(n)$. The process leading to $x(n)$ on the basis of $X(z)$ is known as inversion. This task can be solved by evaluating the complex integral

$$x(n) = \frac{1}{2\pi \mathrm{j}} \oint_C X(z) z^{n-1} \mathrm{d}z \qquad (13.25)$$

around a closed curve c in the *Z*-plane comprising the origin, lying in the region of convergence $X(z)$ and having counter clockwise direction. Formula (13.25) is general but the integral evaluation requires a complex-variable theory.

If $X(z)$ is a rational function of *z*, the inversion can be accomplished by the partial fraction expansion. This approach is very similar to the one used in the Laplace transform. However, a small modification must be introduced as it illustrates the following example.

Example 13.6

Let us consider the inversion of the *Z*-transform

$$X(z) = \frac{z}{(z-0.4)(z-0.8)}.$$

We perform the partial fraction expansion of this function

$$X(z) = \frac{K_1}{z-0.4} + \frac{K_2}{z-0.8}$$

where:

$$K_1 = \lim_{z \to 0.4} (z-0.4) X(z) = -1$$

$$K_2 = \lim_{z \to 2} (z-2) X(z) = 2.$$

Thus, we have

$$X(z) = \frac{-1}{z-0.4} + \frac{2}{z-0.8}.$$

Unfortunately, the two terms on the right hand side do not have a standard form of the *Z*-transform. Therefore, we modify the procedure as follows. We form the function

$$\frac{X(z)}{z}$$

and express it in terms of partial fraction expansion

$$\frac{X(z)}{z} = \frac{1}{(z-0.4)(z-0.8)} = \frac{-2.5}{z-0.4} + \frac{2.5}{z-0.8}.$$

To obtain $X(z)$, we multiply this formula by z

$$X(z) = -2.5\frac{z}{z-0.4} + 2.5\frac{z}{z-0.8}.$$

Since both terms on the right hand side are standard forms, we obtain

$$x(n) = -2.5 \cdot (0.4)^n + 2.5 \cdot (0.8)^n .$$

Another method of inversion exploits the fact that the *Z*-transform is the power series whose coefficients create $x(n)$

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n} = x(0) + x(1) z^{-1} + x(2) z^{-2} + \ldots .$$

Hence, the *Z*-transform should be written as such a series and its coefficients collected to form the sequence $\{x(n)\}$. To obtain the power series, we can directly divide the numerator by the denominator.

Example 13.7

Let the *Z*-transform of a signal be

$$X(z) = \frac{z}{z^2 - 5z + 6} .$$

We arrange the division procedure as follows

$$\begin{array}{l}
z : \left(z^2 - 5z + 6\right) = z^{-1} + 5z^{-2} + 19z^{-3} + \ldots \\
\underline{-z + 5 - 6z^{-1}} \\
\quad 5 - 6z^{-1} \\
\quad \underline{-5 + 25z^{-1} - 30z^{-2}} \\
\qquad 19z^{-1} - 30z^{-2} \\
\qquad \underline{-19z^{-1} + 95z^{-2} - 114z^{-1}} \\
\qquad\qquad 65z^{-2} - 114z^{-1}
\end{array} .$$

Thus, the equation

$$X(z) = 0 \cdot z^0 + z^{-1} + 5z^{-2} + 19z^{-3} + \ldots$$

holds and the coefficients creating $x(n)$ form the set

$$\{0, 1, 5, 19, \ldots\}.$$

13.4. Discrete convolution theorem and its applications

Discrete convolution theorem

Let us consider signals $x_1(n)$ and $x_2(n)$ having the Z-transforms $X_1(z)$ and $X_2(z)$, respectively. Let us assume that $x_1(n) = x_2(n) \equiv 0$ for negative n. The Z-transform of the convolution of the signals $x_1(n)$ and $x_2(n)$ is equal to the product of the Z-transforms of these signals

$$Z(x_1(n) * x_2(n)) = X_1(z) X_2(z).$$

Proof

Let us form the product of two Z-transforms $X_1(z)$ and $X_2(z)$ and denote the product by $X(z)$

$$\begin{aligned} &X(z) = X_1(z) X_2(z) = \\ &= \left(x_1(0) + x_1(1) z^{-1} + x_1(2) z^{-2} + \cdots\right)\left(x_2(0) + x_2(1) z^{-1} + x_2(2) z^{-2} + \cdots\right). \end{aligned} \tag{13.26}$$

On the other hand, $X(z)$ can be considered as the Z-transform of a signal $x(n)$; hence, we have

$$X(z) = x(0) + x(1) z^{-1} + x(2) z^{-2} + \cdots. \tag{13.27}$$

We multiply term by term the two series on the right hand side of (13.26) and compare the coefficients of successive powers of z^{-1} with the corresponding coefficients of the expression (13.27):

$$\begin{aligned} &x(0) = x_1(0) x_2(0) \\ &x(1) = x_1(0) x_2(1) + x_1(1) x_2(0) \\ &x(2) = x_1(0) x_2(2) + x_1(1) x_2(1) + x_1(2) x_2(0) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots . \end{aligned}$$

Generally the coefficient $x(n)$ of z^{-n} in $X(z)$ has the form

$$x(n) = \sum_{m=0}^{n} x_1(m) x_2(n-m) \qquad n = 0, 1, 2, \cdots. \tag{13.28}$$

Since equation (13.28) expresses the convolution of the signals $x_1(n)$ and $x_2(n)$, the theorem holds.

Example 13.8

Let us consider the signals $x_1(n)$ and $x_2(n)$, where:

$$x_1(n) = \{1,\ 2,\ 3,\ 2\}$$

$$x_2(n) = \{1,\ 1,\ 1\}$$

and form the convolution of these signals.
We apply the graphical approach described in Section 1.8.2. The procedure leading to the convolution at $n = 1$ is shown in Fig.13.4.

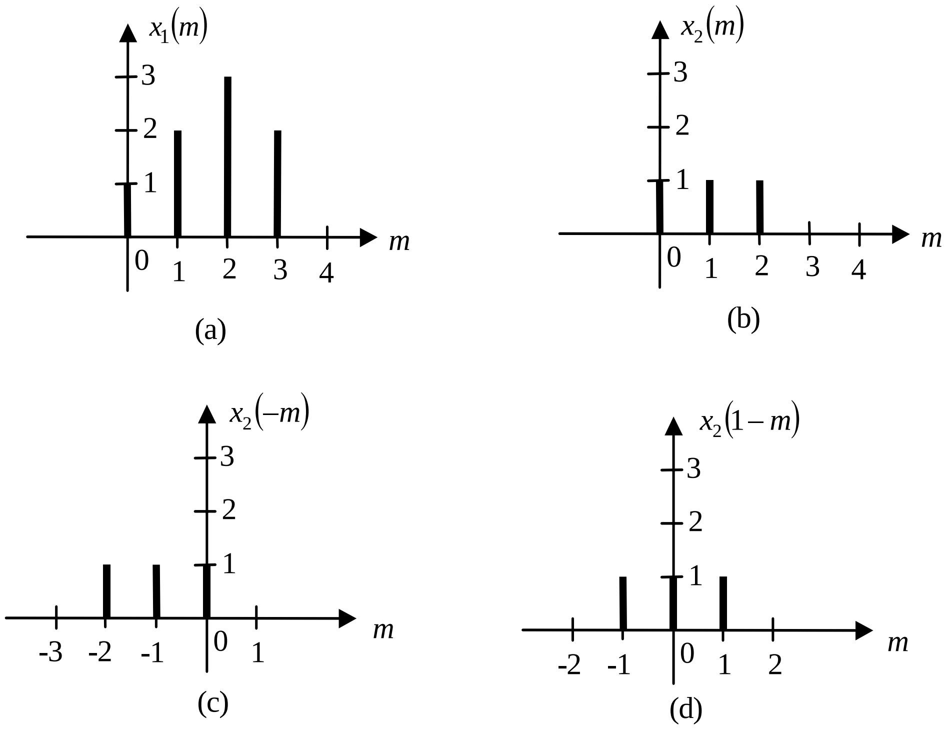

Fig. 13.4. Construction of the convolution at $n = 1$ for Example 13.8

On the basis of Fig.13.4 we obtain $x(1) = 3$. Similarly we find $x(0) = 1$, $x(2) = 6$, $x(3) = 7$, $x(4) = 5$, $x(5) = 2$ and $x(n) = 0$ for $n > 5$.
Thus, the *Z*-transform of the convolution is

$$X(z) = 1 + 3z^{-1} + 6z^{-2} + 7z^{-3} + 5z^{-4} + 2z^{-5}.$$

To verify the discrete convolution theorem, we form the *Z*-transforms of $x_1(n)$ and $x_2(n)$:

$$X_1(z) = 1 + 2z^{-1} + 3z^{-2} + 2z^{-3}$$
$$X_2(z) = 1 + z^{-1} + z^{-2}$$

and multiply $X_1(z)$ by $X_2(z)$, finding, after simple manipulations

$$X_1(z)X_2(z) = 1 + 3z^{-1} + 6z^{-2} + 7z^{-3} + 5z^{-4} + 2z^{-5}.$$

The above result is the same as the one obtained by direct computing the *Z*-transform of the convolution.

A direct corollary of the convolution theorem is the following summation property. If $X(z)$ is the *Z*-transform of $x(n)$, where $x(n) = 0$ for $n < 0$, then the *Z*-transform of the sum $\sum_{k=0}^{n} x(k)$ is $\frac{z}{z-1} X(z)$.

Proof

Let us consider the signal

$$y(n) = x(n) * u(n) = \sum_{k=0}^{n} x(k)u(n-k) = \sum_{k=0}^{n} x(k)$$

where we used the equations $u(n-k) = 0$ for $k > n$ and $u(n-k) = 1$ for $0 \le k \le n$.

Applying the convolution theorem, we obtain

$$Z\left(\sum_{k=0}^{n} x(k)\right) = Y(z) = X(z)U(z) = \frac{z}{z-1} X(z). \tag{13.29}$$

13.5. The transfer function of a discrete system

It is well known that the response of an LTI system to a signal $x(n)$ is given by equation

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

where $h(n)$ is the unit pulse (sample) response of the system. If $x(n) = 0$ for $n < 0$, the lower limit of summation in the above equation reduces to 0 and the equation becomes

$$y(n) = \sum_{k=0}^{\infty} x(k)\, h(n-k)\,. \tag{13.30}$$

Using the discrete convolution theorem to this equation, we obtain

$$Y(z) = H(z)\, X(z)$$

where $X(z)$ is the *Z*-transform of the input signal, $Y(z)$ is the *Z*-transform of the output signal and $\mathrm{H(z)}$ is the *Z*-transform of the LTI system response to the unit pulse input. The ratio $Y(z)/X(z)$ is known as the transfer function of the discrete system. Thus, the transfer function $H(z)$ of any LTI system is equal to the *Z*-transform of the unit pulse response $h(n)$ of this system in which all initial conditions have been set to zero.

Any LTI system can be described in general by the difference equation

$$\begin{aligned} &a_0 y(n) + a_1 y(n-1) + \cdots + a_k y(n-k) = \\ &= b_0 x(n) + b_1 x(n-1) + \cdots + b_\mathrm{p} x(n-\mathrm{p}) \end{aligned} \tag{13.31}$$

where, as above, $x(n)$ is the input signal and $y(n)$ is the output signal. Let us assume that all the initial conditions of the system have been set to zero and take the *Z*-transform of all terms in equation (13.31). Using the shifting property, we obtain

$$\begin{aligned} &a_0 Y(z) + a_1 z^{-1} Y(z) + \cdots + a_k z^{-k} Y(z) = \\ &= b_0 X(z) + b_1 z^{-1} X(z) + \cdots + b_\mathrm{p} z^{-p} X(z) \end{aligned}$$

which can be rearranged to give

$$\frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_p z^{-p}}{a_0 + a_1 z^{-1} + \cdots + a_k z^{-k}}\,. \tag{13.32}$$

The above expression is the transfer function of the discrete system. Thus,

$$H(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_p z^{-p}}{a_0 + a_1 z^{-1} + \cdots + a_k z^{-k}}\,. \tag{13.33}$$

Example 13.9

Let us consider an LTI discrete system described by the difference equation

$$y(n) - 2y(n-1) = 3x(n) \qquad\qquad y(-1) = 0 \tag{13.34}$$

where $x(n)$ is the unit step $u(n)$.
Using the *Z*-transform and the shifting property we have

$$Y(z) - 2\left[z^{-1}Y(z) + y(-1)\right] = 3 \cdot Z(u(n)).$$

Since $y(-1) = 0$ and

$$Z(u(n)) = \frac{z}{z-1}$$

we obtain

$$Y(z) = \frac{3z}{\left(1 - 2z^{-1}\right)(z-1)} = \frac{3z^2}{(z-1)(z-2)}\,.$$

To find $x(n)$, we express $Y(z)/z$ via a partial fraction expansion

$$\frac{Y(z)}{z} = -3\frac{1}{z-1} + 6\frac{1}{z-2}\,.$$

Hence, we have

$$Y(z) = -3\frac{z}{z-1} + 6\frac{z}{z-2}\,.$$

The inversion of the above function is

$$y(n) = -3 + 6 \cdot 2^n .$$

Note that the *Z*-transform of equation (13.34) can be written as follows

$$Y(z) - 2z^{-1}Y(z) = 3X(z) .$$

Thus, the transfer function of the system is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{3}{1-2z^{-1}} = \frac{3z}{z-2} .$$

In Section 4.1 it is shown that a continuous LTI system can be represented by a block diagram composed of blocks, summing points, and pick-off points. Similar approach can be applied to a discrete LTI system. However, in this case a block realizes a multiplication either by a number or a delay (symbolized by z^{-1}). Furthermore, in the summing point the entering signals are added with the plus signs. Therefore, we introduce the sign + inside the circle. The pick-off point functions are as in continuous systems.

To illustrate the block-diagram representation for a discrete system, we consider a system defined by (13.34). The block-diagram for this system is shown in Fig.13.5.

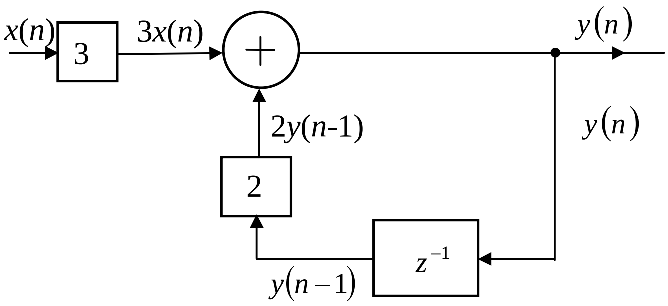

Fig. 13.5. The block-diagram of the system discussed in Example 13.9

13.6. Pole - zero representation for the transfer function

In Section 2 it is shown that the network function of a circuit can be interpreted in terms of its poles and zeros in the complex s plane. A similar concept can be applied to the transfer function which specifies a discrete system. Let us consider the transfer function in the factored form

$$H(z) = K\frac{(z-z_1)(z-z_2)\cdots(z-z_m)}{(z-p_1)(z-p_2)\cdots(z-p_r)} \qquad (13.35)$$

where K is a constant coefficient, $z_1, z_2, \cdots, z_m$ are zeros and $p_1, p_2, \cdots, p_r$ are the poles of this function. Let us assume that $m < r$ and $p_1, \cdots, p_r$ are simple poles. To find the unit sample response of the system, we determine the inverse of $H(z)$ via partial fraction expansion of $H(z)/z$ and multiplication of the result by z

$$H(z) = K_1\frac{z}{z-p_1} + K_2\frac{z}{z-p_2} + \cdots + K_r\frac{z}{z-p_r}\,.$$

The inversion is

$$h(n) = K_1 p_1^n + K_2 p_2^n + \cdots + K_r p_r^n\,.$$

Let us assume that all the poles are real. Then the response has the following properties.
For positive p_k the following holds:

$$\begin{array}{llll} p_k < 1 & p_k^n \to 0 & \text{as} & n \to \infty \\ p_k = 1 & p_k^n = 1 & \text{for} & \text{all} \quad n \\ p_k > 1 & p_k^n \to \infty & \text{as} & n \to \infty\,. \end{array}$$

It p_k is negative, then:

$$\begin{array}{llllll} |p_k| < 1 & p_k^n \to 0 & \text{as} & n \to \infty & \text{and} & p_k^n \text{ alternates in sign} \\ |p_k| = 1 & p_k^n = 1 & \text{for} & n \text{ even} & \text{and} & p_k^n = -1 \quad \text{for} \quad n \quad \text{odd} \\ |p_k| > 1 & |p_k^n| \to \infty & \text{as} & n \to \infty & \text{and} & p_k^n \text{ alternates in sign}\,. \end{array}$$

For complex poles, they must form conjugate pairs and the response corresponding to a pair p_k, p_k^* has the form

$$y(n) = Ap_k^n + A^*\left(p_k^*\right)^n.$$

Setting $A=|A|\mathrm{e}^{\mathrm{j}\alpha}$ and $p_k=|p_k|\mathrm{e}^{\mathrm{j}\theta_k}$, we obtain

$$\begin{aligned} y(n) &= |A||p_k|^n \mathrm{e}^{\mathrm{j}(\alpha+n\theta_k)} + |A||p_k|^n \mathrm{e}^{-\mathrm{j}(\alpha+n\theta_k)} = \\ &= 2|A||p_k|^n \cos(n\theta_k+\alpha). \end{aligned}$$

Since $-1 \le \cos(n\theta_k+\alpha) \le 1$, we conclude that the sequence converges when $|p_k|<1$.

Example 13.10

Let us consider a discrete system specified by a transfer function

$$H(z)=\frac{z}{(z+0.2)(z^2-z+0.5)} .$$

We wish to find the unit sample response of this system. The transfer function has three following poles:

$$\begin{aligned} p_1 &= -0.2 \\ p_2 &= 0.5+\mathrm{j}0.5=\sqrt{0.5}\mathrm{e}^{\mathrm{j}45^\circ} \\ p_3 &= 0.5-\mathrm{j}0.5=\sqrt{0.5}\mathrm{e}^{-\mathrm{j}45^\circ} . \end{aligned}$$

To find the unit sample response, we perform partial fraction expansion of the function $H(z)/z$

$$\frac{H(z)}{z}=K_1\frac{1}{z+0.2}+K_2\frac{1}{z-(0.5+\mathrm{j}0.5)}+K_3\frac{1}{z-(0.5-\mathrm{j}0.5)}$$

where the coefficients K_1, K_2, K_3 are:

$$\begin{aligned} K_1 &= \lim_{z\to-0.2}\frac{H(z)}{z}(z+2)=1.35 \\ K_2 &= \lim_{z\to 0.5+\mathrm{j}0.5}\frac{H(z)}{z}(z-(0.5+\mathrm{j}0.5))=1.16\mathrm{e}^{\mathrm{j}234.5^\circ} \\ K_3 &= K_2^* = 1.16\mathrm{e}^{-\mathrm{j}234.5^\circ} . \end{aligned}$$

Hence, the equation

$$H(z)=1.35\frac{z}{z+0.2}+1.16\mathrm{e}^{\mathrm{j}234.5^\circ}\frac{z}{z-(0.5+\mathrm{j}0.5)}+$$
$$+1.16\mathrm{e}^{-\mathrm{j}234.5^\circ}\frac{z}{z-(0.5-\mathrm{j}0.5)}$$

holds.
Thus, the unit pulse response is

$$y(n)=1.35\cdot(-0.2)^n+1.16\mathrm{e}^{\mathrm{j}234.5^\circ}(0.5+\mathrm{j}0.5)^n+1.16\mathrm{e}^{-\mathrm{j}234.5^\circ}(0.5-\mathrm{j}0.5)^n=$$
$$=1.35\cdot(-0.2)^n+2.32\left(\sqrt{0.5}\right)^n\cos\left(n45^\circ+234.5^\circ\right).$$

Let us consider a causal LTI system represented by the transfer function

$$H(z)=\frac{b_m z^m+b_{m-1}z^{m-1}+\cdots+b_1 z+b_0}{a_r z^r+a_{r-1}z^{r-1}+\cdots+a_1 z+a_0}$$

with $m<r$. Let us assume that the numerator and denominator polynomials do not have any common factor. Recall that the transfer function is the *Z*-transform of the unit pulse response $h(n)$. Hence, $h(n)$ is specified by the poles of the transfer function. If the poles are simple, $h(n)$ includes terms of the form

$$kp^n \qquad\qquad k|p|^n\cos(n\theta+\alpha).$$

Thus, $h(n)\to 0$ as $n\to\infty$ if and only if magnitudes of the poles are less than one, or, in other words, if they are located inside the unit disk of the complex plane. In such a case the unit pulse response is absolutely summable and the system is BIBO stable. It can be shown that the same conclusion is valid for multiple poles.

If any pole is located outside the unit disk (has magnitude greater than one) then the corresponding term of $h(n)$ grows without bound as $n\to\infty$. Consequently $|h(n)|\to\infty$ as $n\to\infty$ and $h(n)$ is not absolutely summable. Therefore, the system is not BIBO stable.

In the special case in which one or more simple poles are located on the unit circle (have the magnitudes equal to one) we say that the system is marginally BIBO stable.

Since the poles of $H(z)$ are zeros of the equations

$$a_r z^r + a_{r-1} z^{r-1} + \cdots + a_1 z + a_0 = 0$$

the system is BIBO stable if all the zeros are located in the open unit disk of the complex plane.

13.7. Solution of difference equations via *Z*-transform

A known fact is that LTI systems are described by linear difference equations with constant coefficients. To solve an equation of this type, we can use the *Z*-transform as follows. We compute the *Z*-transform of each side of the equation and solve the transformed difference equation algebraically for the *Z*-transform of the output signal. Next, we invert the resulting *Z*-transform to find the output signal. To determine the solution of *m*-th order, we must know *m* initial conditions $y(-1), y(-2), \ldots, y(-m)$. These initial conditions are introduced to the expressions for the *Z*-transform of the delayed output signals according to the equation

$$Z(y(n-k)) = y(-k) + y(-k+1)z^{-1} + \cdots + y(-1)z^{-(k-1)} + z^{-k}Y(z). \quad (13.36)$$

The above procedure is illustrated using the following example.

Example 13.11

A discrete LTI system is described by the equation

$$y(n) - y(n-1) - 2y(n-2) = x(n) - 0.1x(n-1)$$

where $x(n) = u(n)$ and the initial conditions are $y(-1) = -1, y(-2) = 0.5$.
To find $y(n)$, we compute the *Z*-transform of this equation

$$\begin{aligned} &Y(z) - \left(y(-1) + z^{-1}Y(z)\right) - 2\left(y(-2) + y(-1)z^{-1} + z^{-2}Y(z)\right) = \\ &= X(z) - 0.1\left(x(-1) + z^{-1}X(z)\right). \end{aligned}$$

Since $x(n) = u(n)$, then $X(z) = \dfrac{z}{z-1}$ and $x(-1) = u(-1) = 0$. Hence, we have

$$\left(1-z^{-1}-2z^{-2}\right)Y(z)-y(-1)-2y(-2)-2y(-1)z^{-1}=$$
$$=\left(1-0.1z^{-1}\right)\frac{z}{z-1}=\frac{z-0.1}{z-1}.$$

Setting the initial conditions and solving for $Y(z)$ yields

$$Y(z)=\frac{-1+1-2z^{-1}+\dfrac{z-0.1}{z-1}}{1-z^{-1}-2z^{-2}}=\frac{-2.1+2z^{-1}+z}{(z-1)\left(1-z^{-1}-2z^{-2}\right)}=$$
$$=\frac{z\left(z^2-2.1z+2\right)}{(z-1)(z+1)(z-2)}.$$

To compute the inverse *Z*-transform of $Y(z)$, we arrange the partial fraction expansion of $Y(z)/z$

$$\frac{Y(z)}{z}=\frac{z^2-2.1z+2}{(z+1)(z-1)(z-2)}=\frac{K_1}{z+1}+\frac{K_2}{z-1}+\frac{K_3}{z-2}$$

where:

$$K_1=\lim_{z\to-1}(z+1)\frac{Y(z)}{z}=\lim_{z\to-1}\frac{z^2-2.1z+2}{(z-1)(z-2)}=0.85$$
$$K_2=\lim_{z\to1}(z-1)\frac{Y(z)}{z}=\lim_{z\to1}\frac{z^2-2.1z+2}{(z+1)(z-2)}=-0.45$$
$$K_3=\lim_{z\to2}(z-2)\frac{Y(z)}{z}=\lim_{z\to2}\frac{z^2-2.1z+2}{(z+1)(z-1)}=0.6\ .$$

Therefore, we may write

$$Y(z)=0.85\frac{z}{z+1}-0.45\frac{z}{z-1}+0.6\frac{z}{z-2}$$

and the output signal $\mathrm{y(n)}$ is

$$y(n)=0.85(-1)^n-0.45u(n)+0.6\cdot2^n\qquad n=0,1,\ldots.$$

The block-diagram which represents the discrete system characterized by the difference equation considered in this example is depicted in Fig.13.6.

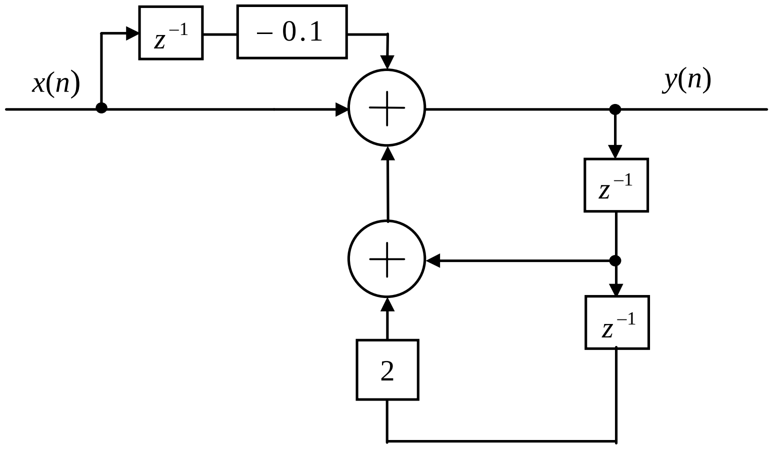

Fig. 13.6. The block-diagram of the system considered in Example 13.11