10. The Discrete-Time Fourier Transform (DTFT)

10.1. Definition of the discrete-time Fourier transform

The Fourier representation of signals plays an important role in both continuous and discrete signal processing. In this section we consider discrete signals and develop a Fourier transform for these signals called the discrete-time Fourier transform, abbreviated DTFT.

The discrete-time Fourier transform of a discrete sequence $x(m)$ is defined as follows:

$$X\left(e^{\mathrm{j}\tilde{\omega}}\right) = \sum_{m=-\infty}^{\infty} x(m)\,\mathrm{e}^{-\mathrm{j}m\tilde{\omega}} \tag{10.1}$$

where $\tilde{\omega}$ is called the normalized frequency.
The notation $X\left(\mathrm{e}^{\mathrm{j}\tilde{\omega}}\right)$ is justified by the observation that the frequency dependency is in exponential form.
In order for the DTFT of a sequence to exist, the summation in (10.1) must converge. It will hold if $x(m)$ is absolutely summable, that is

$$\sum_{m=-\infty}^{\infty} |\, x(m)| < \infty \,. \tag{10.2}$$

Note that the DTFT of a discrete-time sequence is a function of a continuous variable $\tilde{\omega}$.

Since

$$X\left(e^{j(\tilde{\omega}+2\pi)}\right)=\sum_{m=-\infty}^{\infty}x(m)\,e^{-jm(\tilde{\omega}+2\pi)}=\sum_{m=-\infty}^{\infty}x(m)\,e^{-jm\tilde{\omega}}e^{-jm2\pi}=$$

$$=\sum_{m=-\infty}^{\infty}x(m)\,e^{-jm\tilde{\omega}}=X\left(e^{j\tilde{\omega}}\right)$$

then the DTFT is periodic in $\tilde{\omega}$ with a period of 2π.

Since $X\left(e^{j\tilde{\omega}}\right)$ is periodic in $\tilde{\omega}$ with the period equal to 2π, we can express it by an exponential Fourier series in variable $\tilde{\omega}$. Therefore

$$X\left(e^{j\tilde{\omega}}\right)=\sum_{m=-\infty}^{\infty}\tilde{c}_m e^{jm\tilde{\omega}}=\sum_{m=-\infty}^{\infty}\tilde{c}_{-m}e^{-jm\tilde{\omega}} \tag{10.3}$$

where

$$\tilde{c}_{-m}=\frac{1}{2\pi}\int_{-\pi}^{\pi}X\left(e^{j\tilde{\omega}}\right)e^{jm\tilde{\omega}}d\tilde{\omega}. \tag{10.4}$$

Comparison of equations (10.1) and (10.3) shows that the discrete signal $x(m)$ corresponding to the spectrum $X\left(e^{j\tilde{\omega}}\right)$ is $x(m)=\tilde{c}_{-m}$. Therefore, from equation (10.4) the inverse discrete-time Fourier transform is

$$x(m)=\frac{1}{2\pi}\int_{-\pi}^{\pi}X\left(e^{j\tilde{\omega}}\right)e^{jm\tilde{\omega}}d\tilde{\omega}. \tag{10.5}$$

To shed more light on the subject we will derive the DTFT using an alternative approach.

Let us consider the Fourier transform of a continuous-time signal $x(t)$

$$X(j\omega)=\int_{-\infty}^{\infty}x(t)\,e^{-j\omega t}dt\ .$$

We approximate the integral as follows:

$$X(\mathrm{j}\omega) \cong \sum_{m=-\infty}^{\infty} x(mT_s)\,\mathrm{e}^{-\mathrm{j}\omega m T_s} T_s \tag{10.6}$$

and replace the product ωT_s by the normalized frequency

$$\tilde{\omega} = \omega T_s = \frac{\omega}{f_s}\,.$$

If we ignore the scale factor T_s and replace $x(mT_s)$ by $x(m)$, the resulting summation, denoted by $X(\mathrm{e}^{\mathrm{j}\tilde{\omega}})$, is the DTFT

$$X(\mathrm{e}^{\mathrm{j}\tilde{\omega}}) = \sum_{m=-\infty}^{\infty} x(m)\,\mathrm{e}^{-\mathrm{j}\tilde{\omega} m}\,. \tag{10.7}$$

Example 10.1

Let us consider the signal

$$x(m) = a^m u(m) \qquad\qquad |a| < 1\,.$$

The DTFT of this signal is

$$X(\mathrm{e}^{\mathrm{j}\tilde{\omega}}) = \sum_{m=0}^{\infty} a^m \mathrm{e}^{-\mathrm{j}m\tilde{\omega}} = \sum_{m=0}^{\infty} \left(a\mathrm{e}^{-\mathrm{j}\tilde{\omega}}\right)^m\,.$$

Since the expression on the right hand side is the geometric series, we obtain

$$X(\mathrm{e}^{\mathrm{j}\tilde{\omega}}) = \frac{1}{1 - a\mathrm{e}^{-\mathrm{j}\tilde{\omega}}}$$

provided $|a| < 1$.

Example 10.2

Let $x(m)$ be the unit sample $\delta(m)$. The DTFT of this signal is

$$X(\mathrm{e}^{\mathrm{j}\tilde{\omega}}) = \sum_{m=-\infty}^{\infty} \delta(m)\mathrm{e}^{-\mathrm{j}\tilde{\omega} m} = 1\,.$$

10.2. Some properties of the DTFT

In this section we formulate some properties of the discrete time Fourier transform.

Periodicity

This property has already been considered and it can be written as follows

$$X\left(e^{j(\tilde{\omega}+2\pi)}\right) = X\left(e^{j\tilde{\omega}}\right). \tag{10.8}$$

Linearity

The DTFT is a linear operator, i.e. the discrete-time Fourier transform of a signal

$$x(m) = a_1 x_1(m) + a_2 x_2(m)$$

is

$$X\left(e^{j\tilde{\omega}}\right) = a_1 X_1\left(e^{j\tilde{\omega}}\right) + a_2 X_2\left(e^{j\tilde{\omega}}\right)$$

where $X_k\left(e^{j\tilde{\omega}}\right)$ is the DTFT of $x_k(m)$ $(k = 1, 2)$.

Shifting

Let us consider a shifted signal

$$\hat{x}(m) = x(m - m_0).$$

The DTFT of this signal is (see (10.7))

$$\hat{X}\left(e^{j\tilde{\omega}}\right) = \sum_{m=-\infty}^{\infty} x(m - m_0)\, e^{-jm\tilde{\omega}} = \sum_{m=-\infty}^{\infty} x(m - m_0)\, e^{-j(m-m_0)\tilde{\omega}} e^{-jm_0\tilde{\omega}}.$$

Let $k = m - m_0$, then

$$\hat{X}\left(e^{j\tilde{\omega}}\right) = e^{-jm_0\tilde{\omega}} \sum_{k=-\infty}^{\infty} x(k)\, e^{-jk\tilde{\omega}} = e^{-jm_0\tilde{\omega}} X\left(e^{j\tilde{\omega}}\right).$$

Thus, we conclude that shifting in time results in the multiplication of the DTFT by a complex exponential $e^{-jm_0\tilde{\omega}}$.

Example 10.3

Let us consider the shifted unit sample $x(m)=\delta(m-m_0)$. Using the shifting property and knowing that the DTFT of $\delta(m)$ is 1, we obtain

$$X\left(e^{j\tilde{\omega}}\right)=e^{-jm_0\tilde{\omega}}.$$

Frequency shifting

Let us consider a signal $x(m)$ multiplied by $e^{jm\omega_0}$

$$\hat{x}(m)=x(m)\,e^{jm\omega_0}.$$

The DTFT of this signal is

$$\hat{X}\left(e^{j\tilde{\omega}}\right)=\sum_{m=-\infty}^{\infty}x(m)\,e^{jm\omega_0}\,e^{-jm\tilde{\omega}}=\sum_{m=-\infty}^{\infty}x(m)\,e^{-jm(\tilde{\omega}-\omega_0)}=X\left(e^{j(\tilde{\omega}-\omega_0)}\right).$$

Thus, multiplying a sequence by a complex exponential $e^{jm\omega_0}$ results in shifting in frequency of the DTFT.

Convolution theorem

The convolution of signals $x(m)$ and $y(m)$ is given by

$$x(m)*y(m)=\sum_{k=-\infty}^{\infty}x(k)\,y(m-k).$$

The DTFT of the convolution is

$$\sum_{m=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty}x(k)\,y(m-k)\right)e^{-jm\tilde{\omega}}=\sum_{k=-\infty}^{\infty}\left(x(k)\sum_{m=-\infty}^{\infty}y(m-k)\,e^{-jm\tilde{\omega}}\right)=$$

$$=\sum_{k=-\infty}^{\infty}\left(x(k)\sum_{m=-\infty}^{\infty}y(m-k)\,e^{-j(m-k)\tilde{\omega}}e^{-jk\tilde{\omega}}\right)=\sum_{k=-\infty}^{\infty}\left(x(k)\,e^{-jk\tilde{\omega}}\sum_{p=-\infty}^{\infty}y(p)\,e^{-jp\tilde{\omega}}\right)=$$

$$=X\left(e^{j\tilde{\omega}}\right)Y\left(e^{j\tilde{\omega}}\right)$$

where $p = m - k$. Thus, the DTFT of a convolution of signals $x(m)$ and $y(m)$ is the product of the DTFTs of $x(m)$ and $y(m)$.

Parseval's theorem

Let us consider a discrete signal $x(m)$. Parseval's theorem states that

$$\sum_{m=-\infty}^{\infty} |\, x(m)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left| X\left(e^{j\tilde{\omega}}\right)\right|^2 d\tilde{\omega} . \qquad (10.9)$$

In Section 12 it will be shown that this equation gives signal energy in the time domain and in the frequency domain.

10.3. Comparing of the DTFT to the DFT

Recall that the DFT of a sequence $\{x_m = x(m)\}$ where $m = 0,1,\cdots, N-1$ is

$$X_n = \sum_{m=0}^{N-1} x_m w^{-mn} = \sum_{m=0}^{N-1} x_m e^{-j\frac{2\pi}{N}mn} \qquad n = 0,1,\cdots,\cdots N-1 . \qquad (10.10)$$

On the other hand, the DTFT of the same sequence is

$$X\left(e^{j\tilde{\omega}}\right) = \sum_{m=0}^{N-1} x_m e^{-jm\tilde{\omega}} . \qquad (10.11)$$

Comparing (10.10) to (10.11) we find

$$X_n = X\left(e^{j\tilde{\omega}}\right)\Big|_{\tilde{\omega}=\frac{2\pi n}{N}} \qquad n = 0,1,\cdots,\cdots N-1 . \qquad (10.12)$$

Equation (10.12) states that the coefficients of the DFT are samples of the continuous spectrum given by the DTFT at $\tilde{\omega} = \frac{2\pi}{N} n$.

Note that the DFT coefficients correspond to N samples of the $X(z)$ taken at N equally spaced points around the unit circle

$$z = \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}n} \qquad\qquad n = 0,1,\cdots,N-1 .$$

10.4. Generalized DTFT

Some discrete-time signals do not have a DTFT but they have a generalized DTFT as explained below.

Let the DTFT of a signal $x(m)$ be $X\left(\mathrm{e}^{\mathrm{j}\tilde{\omega}}\right) = \delta(\tilde{\omega})$. To find this signal, we use the inverse DTFT:

$$x(m) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\delta(\tilde{\omega})\mathrm{e}^{\mathrm{j}m\tilde{\omega}}\mathrm{d}\tilde{\omega} = \frac{1}{2\pi} .$$

This result states that the constant signal $x(m) = \frac{1}{2\pi}$ has the DTFT equal to $\delta(\tilde{\omega})$. Hence, the constant signal $x(m) = 1$ has the DTFT equal to $2\pi\delta(\tilde{\omega})$, or

$$x(m) = 1 \quad \leftrightarrow \quad X\left(\mathrm{e}^{\mathrm{j}\tilde{\omega}}\right) = 2\pi\delta(\tilde{\omega}) . \tag{10.13}$$

Note that the signal $x(m) = 1$ does not have the DTFT in the ordinary sense because the series

$$\sum_{m=-\infty}^{\infty}\mathrm{e}^{-\mathrm{j}m\tilde{\omega}}$$

is not convergent. Therefore we say that $2\pi\delta(\tilde{\omega})$ is a generalized DTFT of the signal $x(m) = 1$.

Now we consider a discrete signal $x(m)$ having the DTFT $X\left(\mathrm{e}^{\mathrm{j}\tilde{\omega}}\right) = \delta(\tilde{\omega} + \omega_0)$. Then

$$x(m) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\delta(\tilde{\omega} + \omega_0)\mathrm{e}^{\mathrm{j}m\tilde{\omega}}\mathrm{d}\tilde{\omega} = \frac{1}{2\pi}\mathrm{e}^{-\mathrm{j}m\omega_0}$$

or

$$x(m) = \mathrm{e}^{-\mathrm{j}m\omega_0} \quad \leftrightarrow \quad X\left(\mathrm{e}^{\mathrm{j}\tilde{\omega}}\right) = 2\pi\delta(\tilde{\omega} + \omega_0) \tag{10.14}$$

holds.

Likewise, we find

$$x(m) = \mathrm{e}^{\mathrm{j}m\omega_0} \quad \leftrightarrow \quad X\left(\mathrm{e}^{\mathrm{j}\tilde{\omega}}\right) = 2\pi\delta(\tilde{\omega} - \omega_0). \tag{10.15}$$

Since

$$cos(\omega_0 m + \alpha) = \frac{1}{2}\mathrm{e}^{\mathrm{j}(\omega_0 m + \alpha)} + \frac{1}{2}\mathrm{e}^{-\mathrm{j}(\omega_0 m + \alpha)} =$$
$$= \frac{1}{2}\mathrm{e}^{\mathrm{j}\alpha}\mathrm{e}^{\mathrm{j}\omega_0 m} + \frac{1}{2}\mathrm{e}^{-\mathrm{j}\alpha}\mathrm{e}^{-\mathrm{j}\omega_0 m}$$

then using (10.14) and (10.15) we determine the DTFT of the signal $x(m) = \cos(\omega_0 m + \alpha)$ as follows:

$$X\left(\mathrm{e}^{\mathrm{j}\tilde{\omega}}\right) = \frac{1}{2}\mathrm{e}^{\mathrm{j}\alpha}\,2\pi\delta(\tilde{\omega} - \omega_0) + \frac{1}{2}\mathrm{e}^{-\mathrm{j}\alpha}\,2\pi\delta(\tilde{\omega} + \omega_0) =$$
$$= \pi\left(\mathrm{e}^{\mathrm{j}\alpha}\delta(\tilde{\omega} - \omega_0) + \mathrm{e}^{-\mathrm{j}\alpha}\delta(\tilde{\omega} + \omega_0)\right)$$

or

$$x(m) = \cos(\omega_0 m + \alpha) \leftrightarrow X\left(\mathrm{e}^{\mathrm{j}\tilde{\omega}}\right) = \pi\left(\mathrm{e}^{\mathrm{j}\alpha}\delta(\tilde{\omega} - \omega_0) + \mathrm{e}^{-\mathrm{j}\alpha}\delta(\tilde{\omega} + \omega_0)\right). \tag{10.16}$$

A similar approach leads to the DTFT of the signal $x(m) = \sin(\omega_0 m + \alpha)$

$$x(m) = \sin(\omega_0 m + \alpha) = \frac{1}{2\mathrm{j}}\mathrm{e}^{\mathrm{j}(\omega_0 m + \alpha)} - \frac{1}{2\mathrm{j}}\mathrm{e}^{-\mathrm{j}(\omega_0 m + \alpha)} =$$
$$= \frac{1}{2\mathrm{j}}\mathrm{e}^{\mathrm{j}\alpha}\mathrm{e}^{\mathrm{j}\omega_0 m} - \frac{1}{2\mathrm{j}}\mathrm{e}^{-\mathrm{j}\alpha}\mathrm{e}^{-\mathrm{j}\omega_0 m}.$$

To determine the DTFT of $x(m)$, we apply (10.14) and (10.15)

$$X\left(\mathrm{e}^{\mathrm{j}\tilde{\omega}}\right) = \frac{1}{2\mathrm{j}}\mathrm{e}^{\mathrm{j}\alpha}\,2\pi\delta(\tilde{\omega} - \omega_0) - \frac{1}{2\mathrm{j}}\mathrm{e}^{-\mathrm{j}\alpha}\,2\pi\delta(\tilde{\omega} + \omega_0) =$$
$$= \mathrm{j}\pi\left(\mathrm{e}^{-\mathrm{j}\alpha}\delta(\tilde{\omega} + \omega_0) - \mathrm{e}^{\mathrm{j}\alpha}\delta(\tilde{\omega} - \omega_0)\right)$$

or

$$x(m) = \sin(\omega_0 m + \alpha) \leftrightarrow X\left(\mathrm{e}^{\mathrm{j}\tilde{\omega}}\right) = \mathrm{j}\pi\left(\mathrm{e}^{-\mathrm{j}\alpha}\delta(\tilde{\omega} + \omega_0) - \mathrm{e}^{\mathrm{j}\alpha}\delta(\tilde{\omega} - \omega_0)\right). \tag{10.17}$$

In the special case when $\alpha = 0$ we have:

$$x(m) = \cos\omega_0 m \quad \leftrightarrow \quad X\left(e^{j\tilde{\omega}}\right) = \pi\left(\delta(\tilde{\omega} - \omega_0) + \delta(\tilde{\omega} + \omega_0)\right) \tag{10.18}$$

$$x(m) = \sin\omega_0 m \quad \leftrightarrow \quad X\left(e^{j\tilde{\omega}}\right) = j\pi\left(\delta(\tilde{\omega} + \omega_0) - \delta(\tilde{\omega} - \omega_0)\right). \tag{10.19}$$

10.5. Frequency response of LTI discrete systems

Let an LTI discrete system be represented by its unit sample response $h(m)$ (see Fig.10.1). The response of the system due to the input $x(m)$ is given by convolution

$$y(m) = h(m) * x(m) = \sum_{k=-\infty}^{\infty} h(k)x(m-k).$$

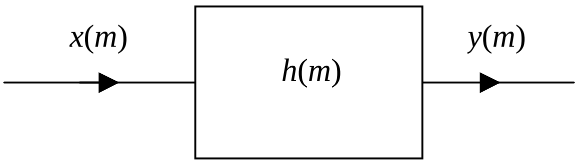

Fig.10.1. An LTI discrete system

Convolution theorem states that

$$Y\left(e^{j\tilde{\omega}}\right) = H\left(e^{j\tilde{\omega}}\right)X\left(e^{j\tilde{\omega}}\right) \tag{10.20}$$

where $Y\left(e^{j\tilde{\omega}}\right)$ is the DTFT of the output $y(m)$, $X\left(e^{j\tilde{\omega}}\right)$ is the DTFT of the input $x(m)$ and $H\left(e^{j\tilde{\omega}}\right)$ is called the frequency response function of the discrete system.

Note that $H\left(e^{j\tilde{\omega}}\right)$ is, in general, a complex-valued function of the frequency $\tilde{\omega}$ and can be written in the polar representation

$$H\left(e^{j\tilde{\omega}}\right) = \left|H\left(e^{j\tilde{\omega}}\right)\right| e^{j\phi_H\left(e^{j\tilde{\omega}}\right)}. \tag{10.21}$$

Thus, the frequency response function of the discrete system is the DTFT of the unit sample response $h(m)$ and is a continuous function of $\tilde{\omega}$. Having the magnitude $\left|H\left(e^{j\tilde{\omega}}\right)\right|$ and the phase $\phi_H\left(e^{j\tilde{\omega}}\right)$ we find:

$$\left|Y\left(e^{j\tilde{\omega}}\right)\right| = \left|H\left(e^{j\tilde{\omega}}\right)\right|\left|X\left(e^{j\tilde{\omega}}\right)\right| \tag{10.22}$$

$$\phi_Y\left(e^{j\tilde{\omega}}\right) = \phi_H\left(e^{j\tilde{\omega}}\right) + \phi_X\left(e^{j\tilde{\omega}}\right). \tag{10.23}$$

Example 10.4

Let us consider the input signal

$$x(m) = A\cos\left(\omega_0 m + \alpha\right)$$

To find the output response of a system specified by a frequency response function $H\left(e^{j\tilde{\omega}}\right)$, we apply (10.20). The DTFT of the signal $x(m)$ is given by (10.16) repeated below

$$X\left(e^{j\tilde{\omega}}\right) = \pi A\left(e^{j\alpha}\delta(\tilde{\omega} - \omega_0) + e^{-j\alpha}\delta(\tilde{\omega} + \omega_0)\right).$$

Hence, we obtain

$$\begin{aligned} Y\left(e^{j\tilde{\omega}}\right) &= \pi A H\left(e^{j\tilde{\omega}}\right) e^{j\alpha}\delta(\tilde{\omega} - \omega_0) + \pi A H\left(e^{j\tilde{\omega}}\right) e^{-j\alpha}\delta(\tilde{\omega} + \omega_0) = \\ &= \pi A\left(e^{j\alpha} H(e^{j\omega_0})\delta(\tilde{\omega} - \omega_0) + e^{-j\alpha} H(e^{-j\omega_0})\delta(\tilde{\omega} + \omega_0)\right) = \\ &= \pi A\left(\left|H\left(e^{j\omega_0}\right)\right|\delta(\tilde{\omega} - \omega_0) e^{j(\phi_H + \alpha)} + \left|H\left(e^{j\omega_0}\right)\right|\delta(\tilde{\omega} + \omega_0) e^{-j(\phi_H + \alpha)}\right). \end{aligned}$$

To find $y(m)$, we use the inverse DTFT

$$\begin{aligned} y(m) &= \frac{A\pi}{2\pi}\int_{-\pi}^{\pi}\left(\left|H\left(e^{j\omega_0}\right)\right|(\delta(\tilde{\omega} - \omega_0) e^{j(\phi_H + \alpha)} + \delta(\tilde{\omega} + \omega_0) e^{-j(\phi_H + \alpha)})\right) e^{jm\tilde{\omega}} d\tilde{\omega} = \\ &= \frac{A}{2}\left|H\left(e^{j\omega_0}\right)\right|\left(e^{j(\omega_0 m + \phi_H + \alpha)} + e^{-j(\omega_0 m + \phi_H + \alpha)}\right) = \\ &= A\left|H\left(e^{j\omega_0}\right)\right|\cos\left(\omega_0 m + \phi_H + \alpha\right). \end{aligned}$$